

LIPSCHITZ EQUIVALENCE OF SELF-SIMILAR SETS WITH TOUCHING STRUCTURES

HUO-JUN RUAN, YANG WANG, AND LI-FENG XI

ABSTRACT. Lipschitz equivalence of self-similar sets is an important area in the study of fractal geometry. It is known that two dust-like self-similar sets with the same contraction ratios are always Lipschitz equivalent. However, when self-similar sets have touching structures the problem of Lipschitz equivalence becomes much more challenging and intriguing at the same time. So far the only known results only cover self-similar sets in \mathbb{R} with no more than 3 branches. In this study we establish results for the Lipschitz equivalence of self-similar sets with touching structures in \mathbb{R} with arbitrarily many branches. Key to our study is the introduction of a geometric condition for self-similar sets called *substitutable*.

1. INTRODUCTION

1.1. **Motivation.** A fundamental concept in fractal geometry is dimension. It is often used to differentiate fractal sets, and when two sets have different dimensions (Hausdorff dimensions, box dimensions or other dimensions) we often consider them to be not alike. However two compact sets, even with the same dimension, may in fact be quite different in many ways. Thus it is natural to seek a suitable quality that would allow us to tell whether two fractal sets are similar. And generally, Lipschitz equivalence is thought to be such a suitable quality.

It has been pointed out in [5] that while topology may be regarded as the study of equivalence classes of sets under homeomorphism, fractal geometry is sometimes thought of as the study of equivalence classes under bi-Lipschitz mappings. The more restrictive maps such as isometry tend to lead to poor and rather boring equivalent classes, while the far less restrictive maps such as general continuous maps take us completely out of geometry into the realm of pure topology (see [6]). Bi-Lipschitz maps offer a good balance, which lead to categories that are interesting and intriguing both geometrically and algebraically.

There are many works done in the field of Lipschitz equivalence of two fractal sets. Some earlier fundamental results are obtained by Cooper and Pignataro [1], David and Semmes [2], and Falconer and Marsh [4, 5]. Recently, based on these works and motivated by Problem 11.16 in [2], Rao, Ruan, Wang, Xi, Xiong and their collaborators obtained a series of results, see e.g. [12]-[14], [19]-[23]. There are also some other related works. Xi [18] discussed the nearly Lipschitz equivalence of self-conformal

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sets. Mattila and Saaranen [10] studied the Lipschitz equivalence of Ahlfors-David regular sets. Deng, Wen, Xiong and Xi [3] and Llorente and Mattila [9] discussed the bi-Lipschitz embedding of fractal sets.

Let E and F be two compact subsets of \mathbb{R}^d . A bijection $f : E \rightarrow F$ is said to be *bi-Lipschitz* if there exist two positive constants c and c' such that

$$(1.1) \quad c|x - y| \leq |f(x) - f(y)| \leq c'|x - y|, \quad \forall x, y \in E.$$

E and F are said to be *Lipschitz equivalent*, denoted by $E \sim F$, if there exists a bi-Lipschitz map f from E to F .

We recall some basic notations in fractal geometry. Given a family of similitude $\Phi_i(x)$, $i = 1, \dots, n$, on \mathbb{R}^d , where each Φ_i has contraction ratio ρ_i with $\rho_i < 1$, there exists a unique nonempty compact subset E of \mathbb{R}^d such that $\bigcup_{i=1}^n \Phi_i(E) = E$, see [7]. The set of maps $\{\Phi_i(x), i = 1, \dots, n\}$ is called an *iterated function system* (IFS) and E is called the *attractor*, or the *invariant set*, of the IFS. We also call E a *self-similar set* since every Φ_i is a similitude. If $\Phi_i(E) \cap \Phi_j(E) = \emptyset$ for any distinct i and j , the IFS $\{\Phi_i\}$ is then said to satisfy the *strong separation condition* (SSC), and E is said to be *dust-like*.

Given $\rho_1, \dots, \rho_n \in (0, 1)$ with $\sum_{i=1}^n \rho_i^d < 1$, we call $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ a (separable) *contraction vector* (in \mathbb{R}^d). We denote by $\mathcal{D}(\boldsymbol{\rho})$ the family of all dust-like self-similar sets with contraction vector $\boldsymbol{\rho}$ (here the ambient dimension d is implicitly fixed). The following property is well known, see e.g. [13].

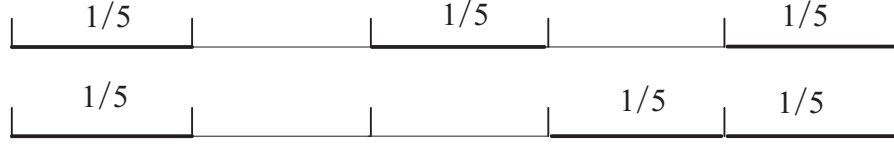
Proposition 1.1. *Any two sets in $\mathcal{D}(\boldsymbol{\rho})$ are Lipschitz equivalent.*

There are examples where two different contraction vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ lead to Lipschitz equivalent families $\mathcal{D}(\boldsymbol{\rho}_1)$ and $\mathcal{D}(\boldsymbol{\rho}_2)$. For example, Rao, Ruan and Wang [12] has completely classified the Lipschitz equivalence of dust-like families $\mathcal{D}(\boldsymbol{\rho})$ where $\boldsymbol{\rho} = (\rho_1, \rho_2)$, and one of the results is that $\boldsymbol{\rho}_1 = (\lambda, \lambda^5)$ and $\boldsymbol{\rho}_2 = (\lambda^2, \lambda^3)$ lead to Lipschitz equivalence families whenever the resulting self-similar sets are dust-like. The paper [12] and some earlier studies such as [1, 5] have explored the impact of algebraic properties of the contraction vectors on Lipschitz equivalence, yielding a number of intriguing results showing the links.

Nevertheless one should not overlook the importance of geometric properties of the underlying IFSs has on Lipschitz equivalence of self-similar sets. Relating to this point is an interesting problem proposed by David and Semmes [2] (Problem 11.16).

Problem 1.1. *Let $S_i(x) := x/5 + (i-1)/5$ be a contractive map from $[0, 1]$ to $[0, 1]$ where $i \in \{1, \dots, 5\}$. Let M and M' be the attractor of the IFS $\{S_1, S_3, S_5\}$ and the IFS $\{S_1, S_4, S_5\}$, respectively. Are M and M' Lipschitz equivalent?*

We call M the $\{1, 3, 5\}$ -set and M' the $\{1, 4, 5\}$ -set. Clearly, M is dust-like and M' has certain touching structure, see Figure 1. In this problem, the contraction ratios are all identical so the difference lies

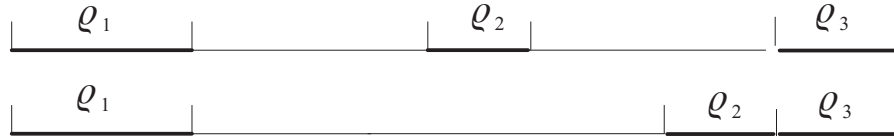
FIGURE 1. Initial construction of M and M'

entirely in the geometry of the two IFSs. David and Semmes conjectured that $M \not\sim M'$. However, by examining graph-directed structures of the attractors and introducing techniques to study Lipschitz equivalence on these structures, Rao, Ruan and Xi [13] proved that in fact $M \sim M'$.

Notice that all contractive maps in above problem have same contraction ratio $1/5$. Some similar works has been done in higher dimensional case, e.g. [8, 16, 17, 22].

A follow up study in Xi and Ruan [20] exploits the interplay of algebraic properties of contraction vectors and geometric properties of IFSs. It considers the following generalization of the $\{1, 3, 5\} - \{1, 4, 5\}$ problem:

Problem 1.2. Let $\rho = (\rho_1, \rho_2, \rho_3)$ be a contraction vector (in \mathbb{R}). Let $\Phi_i(x) = \rho_i x + d_i$, $i \in \{1, 2, 3\}$, where $d_1 = 0, d_3 = 1 - \rho_3$ and $\rho_1 < d_2 < 1 - \rho_2 - \rho_3$ (e.g. $d_2 = \rho_1 + (1 - \rho_1 - \rho_2 - \rho_3)/2$). Let $\Psi_1 = \Phi_1$, $\Psi_3 = \Phi_3$ and $\Psi_2(x) = \rho_2 x + t_2$ with $t_2 = 1 - \rho_2 - \rho_3$. Let M_ρ and M'_ρ be the attractor of $\{\Phi_1, \Phi_2, \Phi_3\}$ and $\{\Psi_1, \Psi_2, \Psi_3\}$, respectively. See Figure 2 for their initial configuration. Are M_ρ and M'_ρ Lipschitz equivalent?

FIGURE 2. Initial construction of M_ρ and M'_ρ

Somewhat surprisingly, the Lipschitz equivalence of the two sets are completely determined by the algebraic property of ρ_1 and ρ_3 and independent of ρ_2 . It is shown in [20] that $M_\rho \sim M'_\rho$ if and only if $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$.

The above example is nevertheless a very special case. It is natural to exploit such algebraic and geometric connections further in more general settings, which is the aim of this paper. Given the complexity of even to establish the result for Problem 1.2, this may appears to be a very daunting task. Fortunately, by introducing a new geometric notion called *substitutable* we are able to prove a number of results in this direction.

Throughout this paper we assume that $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ is a contraction vector (in \mathbb{R}) with $n \geq 3$. Let $D \in \mathcal{D}(\boldsymbol{\rho})$. By Proposition 1.1, we may assume without loss of generality that D is the attractor of the IFS $\{\Phi_i(x) = \rho_i x + d_i\}_{i=1}^n$, where $\Phi_1([0, 1]), \dots, \Phi_n([0, 1])$ are equally spaced closed subintervals of $[0, 1]$ arranged from left to right, normalized so that the left endpoint of $\Phi_1([0, 1])$ is 0 and the right end of $\Phi_n([0, 1])$ is 1.

We are interested in the Lipschitz equivalence of D with the attractor T of another IFS $\{\Psi_i(x) = \rho_i x + t_i\}_{i=1}^n$ having the same contraction vector $\boldsymbol{\rho}$ but with translations $\{t_i\}$ that may result in some of the subintervals $\Psi_1([0, 1]), \dots, \Psi_n([0, 1])$ touching one another (but no overlapping). More precisely, the IFS $\{\Psi_i(x) = \rho_i x + t_i\}_{i=1}^n$ satisfies the following three properties:

- (1) The subintervals $\Psi_1([0, 1]), \dots, \Psi_n([0, 1])$ are spaced from left to right without overlapping, i.e. their interiors do not intersect.
- (2) The left endpoint of $\Psi_1[0, 1]$ is 0 and the right endpoint of $\Psi_n[0, 1]$ is 1.
- (3) There exists at least one $i \in \{1, 2, \dots, n-1\}$, such that the intervals $\Psi_i([0, 1])$ and $\Psi_{i+1}([0, 1])$ are touching, i.e. $\Psi_i(1) = \Psi_{i+1}(0)$.

Denote by T the attractor of the IFS $\{\Psi_i\}_{i=1}^n$. Figure 3 gives an example of $\{\Phi_i\}$ and $\{\Psi_i\}$, respectively. In this paper we present necessary conditions and sufficient conditions for $D \sim T$.

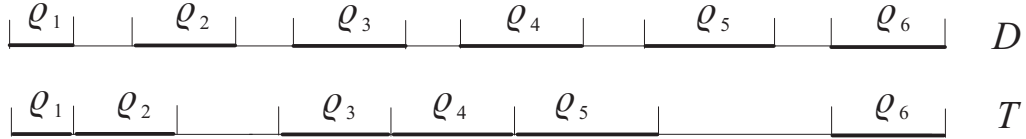


FIGURE 3. Initial construction of D and T , where $n = 6$

1.2. Notations and Examples. First some commonly used basic notations. Denote $\Sigma_n := \{1, 2, \dots, n\}$ and $\Sigma_n^* := \bigcup_{m \geq 1} \Sigma_n^m = \bigcup_{m \geq 1} \{1, 2, \dots, n\}^m$. We shall call any $i \in \Sigma_n$ a *letter* and $\mathbf{i} = i_1 \dots i_m \in \Sigma_n^*$ a *word* of length $|\mathbf{i}| := m$. i_1 and i_m is called the *first letter* and *last letter* of \mathbf{i} , respectively. We define $\rho_{\mathbf{i}} = \rho_{i_1} \dots \rho_{i_m}$, $\Psi_{\mathbf{i}} = \Psi_{i_1} \circ \dots \circ \Psi_{i_m}$ and $T_{\mathbf{i}} = \Psi_{\mathbf{i}}(T)$. $T_{\mathbf{i}}$ is called a *cylinder* of the IFS $\{\Psi_i\}$ for \mathbf{i} . Similarly we define $\Phi_{\mathbf{i}} = \Phi_{i_1} \circ \dots \circ \Phi_{i_m}$ and the cylinder $D_{\mathbf{i}} = \Phi_{\mathbf{i}}(D)$.

Specific to this study we introduce also other notations. A letter $i \in \Sigma_n$ is a *(left) touching letter* if $\Psi_i([0, 1])$ and $\Psi_{i+1}([0, 1])$ are touching, i.e. $\Psi_i(1) = \Psi_{i+1}(0)$. We use $\Sigma_T \subset \Sigma_n$ to denote the set of all (left) touching letters. Note that one may view $\Sigma_T + 1$ to be the set of all right touching letters. For simplicity we shall drop the word “left” for Σ_T . Let α and β be the number of successive touching intervals among $\Psi_1([0, 1]), \dots, \Psi_n([0, 1])$ at the beginning and at the end, respectively. In other words, $\bigcup_{i=1}^{\alpha} \Psi_i[0, 1]$ and $\bigcup_{i=n-\beta+1}^n \Psi_i[0, 1]$ are intervals, while $\Psi_{\alpha}(1) \neq \Psi_{\alpha+1}(0)$ and $\Psi_{n-\beta}(1) \neq \Psi_{n-\beta+1}(0)$.

Given a cylinder $T_{\mathbf{i}}$ and a nonnegative integer k , we can define respectively the *level $(k+1)$ left touching patch* and the *level $(k+1)$ right touching patch* of $T_{\mathbf{i}}$ to be

$$(1.2) \quad L_k(T_{\mathbf{i}}) = \bigcup_{j=1}^{\alpha} T_{\mathbf{i}[1]^k j}, \quad R_k(T_{\mathbf{i}}) = \bigcup_{j=n-\beta+1}^n T_{\mathbf{i}[n]^k j},$$

where $[\ell]^k$ is defined to be the word $\underbrace{\ell \cdots \ell}_k$ for any $\ell \in \{1, \dots, n\}$, with $\mathbf{i}[1]^k j$ be the concatenation of \mathbf{i} , $[1]^k$ and the letter j (similarly for $\mathbf{i}[n]^k j$). We remark that $L_0(T_{\mathbf{i}}) = \bigcup_{j=1}^{\alpha} T_{\mathbf{i}j}$ and $R_0(T_{\mathbf{i}}) = \bigcup_{j=n-\beta+1}^n T_{\mathbf{i}j}$.

Now comes the main notation we introduce for this paper. A letter $i \in \Sigma_T$ is called *left substitutable* if there exist $\mathbf{j} \in \Sigma_n^*$ and $k, k' \in \mathbb{N}$, such that $\text{diam } L_k(T_{i+1}) = \text{diam } L_{k'}(T_{i\mathbf{j}})$ and the last letter of \mathbf{j} does not belong to $\{1\} \cup (\Sigma_T + 1)$. Geometrically it simply means that certain left touching patch of the cylinder T_{i+1} has the same diameter as that of some left touching patch of a cylinder $T_{i\mathbf{j}}$, and as a result we can substitute one of the left touching patches by the other without disturbing the other neighboring structures in T because they have the same diameter. The actual substitution is performed in the proof of our main theorem. Similarly, $i \in \Sigma_T$ is called *right substitutable* if there exist $\mathbf{j} \in \Sigma_n^*$ and $k, k' \in \mathbb{N}$, such that $\text{diam } R_k(T_i) = \text{diam } R_{k'}(T_{(i+1)\mathbf{j}})$ and the last letter of \mathbf{j} does not belong to $\{n\} \cup \Sigma_T$. We say that $i \in \Sigma_T$ is *substitutable* if it is left substitutable or right substitutable.

Remark 1.1. Both left and right substitutable properties can be characterized algebraically as well. By definition, it is easy to check that $\text{diam } L_k(T_{i+1}) = \text{diam } L_{k'}(T_{i\mathbf{j}})$ is equivalent to

$$(1.3) \quad \rho_{i+1} \rho_1^k = \rho_i \rho_1^{k'} \rho_{\mathbf{j}},$$

while $\text{diam } R_k(T_i) = \text{diam } R_{k'}(T_{(i+1)\mathbf{j}})$ is equivalent to

$$(1.4) \quad \rho_i \rho_n^k = \rho_{i+1} \rho_n^{k'} \rho_{\mathbf{j}}.$$

Example 1.1. Let Ψ_1, Ψ_2, Ψ_3 be defined as in Problem 1.2 and let T be its attractor. Clearly $\Sigma_T = \{2\}$, $\alpha = 1$ and $\beta = 2$. Assume that $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$, i.e. there exist $u, v \in \mathbb{Z}^+$ such that $\rho_1^u = \rho_3^v$. Pick $k = v + 1$, $k' = 0$ and $\mathbf{j} = 2[1]^u$. It is easy to check that (1.4) holds for $i = 2$ and the last letter of \mathbf{j} is $1 \notin \{3\} \cup \Sigma_T$. Thus the touching letter 2 is right substitutable. See Figure 4 for a graphical illustration.

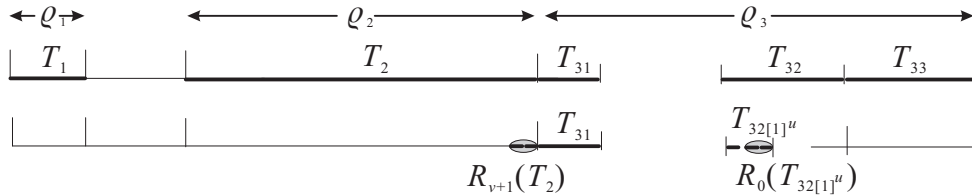


FIGURE 4. The unique touching letter 2 is right substitutable in Example 1.1

1.3. Statement of Results. We establish several results in this paper. First we prove the following necessary condition for $D \sim T$, regardless of the geometric configuration of the IFS $\{\Psi_i\}$:

Theorem 1.1. *Assume that $D \sim T$. Then $\log \rho_1 / \log \rho_n \in \mathbb{Q}$.*

As a result we shall always assume in this paper that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$. For the case of $n = 3$ branches it was shown in [20] that the condition $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$ is also sufficient for the Lipschitz equivalence of D and T . So naturally one may ask whether this condition is sufficient in general. The following theorem shows that this is false, even for the 4-branch case.

Theorem 1.2. *Let $n = 4$, $\rho_1 = \rho_4$, and $\Sigma_T = \{2\}$. Assume that $D \sim T$. Let s be the common Hausdorff dimension of D and T and $\mu_i = \rho_i^s$ for $1 \leq i \leq 4$. Then μ_2 and μ_3 must be algebraically dependent, namely there exists a rational nonzero polynomial $P(x, y)$ such that $P(\mu_2, \mu_3) = 0$.*

Later in the paper we shall see that if $\log \rho_i / \log \rho_j \in \mathbb{Q}$ for all $i, j \in \{1, \dots, n\}$ then $D \sim T$. To go deeper we must take into account the geometric information of the IFS $\{\Psi_i\}$. The main theorem of the paper is:

Theorem 1.3. *Assume that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$. Then, $D \sim T$ if every touching letter for T is substitutable.*

As indicated in Example 1.1, the IFS in question in Problems 1.2 has a single touching letter $\Sigma_T = \{2\}$ and the letter is substitutable. Thus the Lipschitz equivalences in both problems follow directly from Theorem 1.3.

Corollary 1.1 ([20]). *Let $\rho = (\rho_1, \rho_2, \rho_3)$ and M_ρ and M'_ρ be sets defined in Problem 1.2. Then $M_\rho \sim M'_\rho$ if and only if $\log \rho_1 / \log \rho_3 \in \mathbb{Q}$.*

Corollary 1.2 ([13]). *Let M and M' be sets defined in Problem 1.1. Then $M \sim M'$.*

Theorem 1.3 allows us to establish a more general corollary. The argument used to show the substitutability in Example 1.1 is easily extended to prove the following corollary:

Corollary 1.3. *$D \sim T$ if one of the following conditions holds:*

- (1) $\log \rho_i / \log \rho_j \in \mathbb{Q}$ for all $i, j \in \{1, n, \alpha\} \cup (\Sigma_T + 1)$.
- (2) $\log \rho_i / \log \rho_j \in \mathbb{Q}$ for all $i, j \in \{1, n, n - \beta + 1\} \cup \Sigma_T$.

Proof. Without loss of generality, we only prove (1). Given a touching letter i . We will show that i is right substitutable. Since $\log \rho_\alpha / \log \rho_n, \log \rho_{i+1} / \log \rho_n \in \mathbb{Q}$, there exist $u, v, w \in \mathbb{Z}^+$ such that

$\rho_\alpha^u = \rho_n^v = \rho_{i+1}^w$. Pick $k = 2v$, $k' = 0$ and $\mathbf{j} = i[i+1]^{w-1}[\alpha]^u$. It is easy to check that (1.4) holds. Notice that $\alpha \notin \{n\} \cup \Sigma_T$. It follows that i is right substitutable. \blacksquare

The following result, which we wish to state as a theorem because of the simplicity of its statement, is a direct corollary of Corollary 1.3.

Theorem 1.4. *Assume that $\log \rho_i / \log \rho_j \in \mathbb{Q}$ for all $i, j \in \{1, \dots, n\}$. Then $D \sim T$.*

We remark that the above condition is clearly not a necessary condition, as we have seen from the 3-branch case, for which the contraction ratio of the middle branch is irrelevant. One difference between the dust-like case and the touching case is that the order of the contraction ratios do matter, as Theorem 1.1 indicates. However, the condition in Theorem 1.4 can be viewed as a weak necessary condition in the sense that given a set of contraction ratios ρ_1, \dots, ρ_n , if $\log \rho_i / \log \rho_j \notin \mathbb{Q}$ for some $i, j \in \{1, \dots, n\}$ then there exists a touching IFS whose contraction ratios are $\{\rho_i\}$ such that its attractor T is not Lipschitz equivalent to D . This is easily done by making the contraction ratios of the left most and right most branches to be ρ_i and ρ_j , respectively.

The rest of the paper will be devoted to proving the stated results. In Section 2 we prove Theorems 1.1 and 1.2, and in Section 3 we prove Theorem 1.3.

2. NECESSARY CONDITION FOR $D \sim T$

2.1. Bi-Lipschitz map related with a dust-like self-similar set. In this subsection, we will discuss the property of bi-Lipschitz map $f : E \rightarrow F$, where E is a nonempty compact subset of \mathbb{R}^d and F is a dust-like self-similar subset of \mathbb{R}^d with contraction vector (ρ_1, \dots, ρ_n) . We assume that

$$(2.1) \quad c|x - y| \leq |f(x) - f(y)| \leq c'|x - y|, \quad \forall x, y \in E,$$

where $0 < c \leq c'$.

For any nonempty subsets $A, B \subset \mathbb{R}^d$, we define $d(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. The diameter of A is defined to be $\text{diam } A := \sup\{|x - y| : x, y \in A\}$. If $d(A, B \setminus A) > 0$, we say that A is a *B-separate set*. If $d(A, B \setminus A) \geq \lambda \cdot \text{diam } A$ for some $\lambda > 0$, we say that A is a *(B, λ)-separate set*.

We now present a lemma which is similar to [5, Lemma 3.2].

Lemma 2.1. *For any $\lambda > 0$, there exists an integer n_0 such that for any (E, λ) -separate set $A \subset E$, there exist $\mathbf{k}, \mathbf{j}_1, \dots, \mathbf{j}_p \in \Sigma_n^*$ such that $F_{\mathbf{k}\mathbf{j}_1}, \dots, F_{\mathbf{k}\mathbf{j}_p}$ are disjoint and*

$$(2.2) \quad f(A) = \bigcup_{r=1}^p F_{\mathbf{k}\mathbf{j}_r} \subset F_{\mathbf{k}},$$

where each $|\mathbf{j}_r| = n_0$.

Proof. Given an (E, λ) -separate set $A \subset E$. Let $F_{\mathbf{k}}$ be the smallest cylinder containing $f(A)$. Then, it is clear that there exists a positive constant δ dependent only on F such that $\text{diam } F_{\mathbf{k}} \leq \delta \text{diam } f(A)$. For a detailed proof, please see e.g. [5, Lemma 3.1]. Thus, by (2.1), we have

$$\text{diam } F_{\mathbf{k}} \leq \delta \text{diam } f(A) \leq \delta c' \text{diam } A.$$

Let n_0 be the smallest integer satisfying $\bar{\rho}^{n_0} \delta c' \leq \frac{c\lambda}{2}$, where $\bar{\rho} = \max\{\rho_1, \dots, \rho_n\}$. Then, if $|\mathbf{j}'| = n_0$,

$$(2.3) \quad \text{diam } F_{\mathbf{kj}'} \leq \bar{\rho}^{n_0} \text{diam } F_{\mathbf{k}} \leq \bar{\rho}^{n_0} \delta c' \text{diam } A \leq \frac{c\lambda}{2} \text{diam } A.$$

Assume that $F_{\mathbf{kj}'} \cap f(A) \neq \emptyset$, we will prove $F_{\mathbf{kj}'} \subset f(A)$ by showing $d(z, f(E \setminus A)) > 0$ for any $z \in F_{\mathbf{kj}'}$.

Pick $x \in f^{-1}(F_{\mathbf{kj}'} \cap f(A))$, we have

$$(2.4) \quad d(f(x), f(E \setminus A)) \geq d(f(A), f(E \setminus A)) \geq c \cdot d(A, E \setminus A) \geq c\lambda \text{diam } A.$$

Thus, for any $z \in F_{\mathbf{kj}'}$, using (2.3) and (2.4), we have

$$d(z, f(E \setminus A)) \geq d(f(x), f(E \setminus A)) - \text{diam } F_{\mathbf{kj}'} \geq c\lambda \text{diam } A - \frac{c\lambda}{2} \text{diam } A > 0.$$

This completes the proof of the lemma. ■

Remark 2.1. By the proof, we can require that $F_{\mathbf{k}}$ is the smallest cylinder containing $f(A)$. Under this restriction, \mathbf{k} is uniquely determined by A . Consequently, the set $\{\mathbf{j}_1, \dots, \mathbf{j}_p\}$ are also uniquely determined by A and n_0 .

2.2. Construction of (T, λ) -separate sets. Let \emptyset be the empty word. We say that the length of \emptyset is 0. Define $\Psi_{\emptyset} = \Phi_{\emptyset} = \text{id}$, $\rho_{\emptyset} = 1$ and $I_{\emptyset} = [0, 1]$. Let $\mathcal{I}_0 = \{I_{\emptyset}\}$ and $\mathcal{I}_m = \{I_{\mathbf{i}} : |\mathbf{i}| = m\}$ for all positive integers m .

By the definition of T , we know that there exists i such that $\Psi_i(1) = \Psi_{i+1}(0)$. We pick one such i and denote it by i_0 . Without loss of generality, we assume that $\rho_1 \geq \rho_n$. For positive integer k , we define $\tau(k)$ to be the unique positive integer satisfying

$$(2.5) \quad \rho_n^k \rho_1 < \rho_1^{\tau(k)} \leq \rho_n^k.$$

It is clear that $\tau(k) \geq k$ and is increasing with respect to k . We define

$$C^k = R_k(T_{i_0}) \cup L_{\tau(k)}(T_{i_0+1}).$$

We remark that $C^1 \supset C^2 \supset \dots$.

We shall adopt the notation \asymp throughout this paper. Let A be a given index set. Given two sequences of positive real numbers $(a_i)_{i \in A}$ and $(b_i)_{i \in A}$ indexed by A , we denote $(a_i) \asymp (b_i)$ if there exist positive constants c_1, c_2 independent of i such that $c_1 a_i \leq b_i \leq c_2 a_i$ for all $i \in A$. For convenience of statement

in the proofs, we shall often write $a_i \asymp b_i$ for all $i \in A$, or simply $a_i \asymp b_i$ if there is no confusion about the index set.

Lemma 2.2. *There exists $\lambda > 0$, such that C^k is (T, λ) -separate for all k .*

Proof. Notice that $R_k(T_{i_0}) \cap L_{\tau(k)}(T_{i_0+1})$ is a singleton. By (2.5), for all k we have

$$\begin{aligned}
 \text{diam } C^k &= \text{diam } R_k(T_{i_0}) + \text{diam } L_{\tau(k)}(T_{i_0+1}) \\
 &= \rho_{i_0} \rho_n^k \cdot \text{diam} \left(\bigcup_{j=n-\beta+1}^n T_j \right) + \rho_{i_0+1} \rho_1^{\tau(k)} \cdot \text{diam} \left(\bigcup_{j=1}^{\alpha} T_j \right) \\
 (2.6) \quad &\asymp \rho_n^k.
 \end{aligned}$$

On the other hand, it is clear that the distance of C^k and $T \setminus C^k$ equals the minimum of the following two distances: $d(R_k(T_{i_0}), T_{i_0[n]^k(n-\beta)})$ and $d(L_{\tau(k)}(T_{i_0+1}), T_{(i_0+1)[1]^{\tau(k)}(\alpha+1)})$. Since

$$d(R_k(T_{i_0}), T_{i_0[n]^k(n-\beta)}) = d(T_{i_0[n]^k(n-\beta+1)}, T_{i_0[n]^k(n-\beta)}) = \rho_{i_0} \rho_n^k \cdot d(T_{n-\beta+1}, T_{n-\beta}),$$

and similarly,

$$d(L_{\tau(k)}(T_{i_0+1}), T_{(i_0+1)[1]^{\tau(k)}(\alpha+1)}) = \rho_{i_0+1} \rho_1^{\tau(k)} \cdot d(T_{\alpha}, T_{\alpha+1}),$$

we know from (2.5) that $d(C^k, T \setminus C^k) \asymp \rho_n^k$ for all k . Combining this with (2.6), we have $d(C^k, T \setminus C^k) \asymp \text{diam } C^k$ for all k . Hence, there exists $\lambda > 0$ such that $d(C^k, T \setminus C^k) \geq \lambda \cdot \text{diam } C^k$ for any k . This completes the proof. ■

For all $\mathbf{i} \in \Sigma_n^* \cup \{\emptyset\}$ and $k \in \mathbb{Z}^+$, we define

$$C_{\mathbf{i}}^k = \Psi_{\mathbf{i}}(C^k).$$

It is clear that $\text{diam } C_{\mathbf{i}}^k = \rho_{\mathbf{i}} \cdot \text{diam } C^k$ and $d(C_{\mathbf{i}}^k, T \setminus C_{\mathbf{i}}^k) = \rho_{\mathbf{i}} \cdot d(C^k, T \setminus C^k)$. Thus, $C_{\mathbf{i}}^k$ is (T, λ) -separated, where λ is defined as in Lemma 2.2. For any $k \in \mathbb{Z}^+$, we define

$$\mathcal{C}_k = \{C_{\mathbf{i}}^j : |\mathbf{i}| + j = k \text{ where } \mathbf{i} \in \Sigma_n^* \cup \{\emptyset\} \text{ and } j \in \mathbb{Z}^+\}.$$

Lemma 2.3. *For any two distinct sets $A, B \in \mathcal{C}_k$, we have $A \cap B = \emptyset$.*

Proof. Suppose that $A = C_{\mathbf{i}}^k$ and $B = C_{\mathbf{j}}^{\ell}$. It is clear that $A \cap B = \emptyset$ if $k \neq \ell$. Thus, without loss of generality, we assume that $k = \ell$.

Case 1. Assume that $\mathbf{j} = \emptyset$. Let $m = |\mathbf{i}|$. Then $m + k = \ell$ and $m \geq 1$. Notice that

$$C^{\ell} = R_{\ell}(T_{i_0}) \cup L_{\tau(\ell)}(T_{i_0+1}) = \left(\bigcup_{j=n-\beta+1}^n T_{i_0[n]^{\ell}j} \right) \cup \left(\bigcup_{j=1}^{\alpha} T_{(i_0+1)[1]^{\tau(\ell)}j} \right).$$

From $C_{\mathbf{i}}^k \subset T_{\mathbf{i}}$ and $\tau(\ell) \geq \ell > m$, we know that $C_{\mathbf{i}}^k \cap C^{\ell} = \emptyset$ if $\mathbf{i} \notin \{i_0[n]^{m-1}, (i_0+1)[1]^{m-1}\}$.

In case that $\mathbf{i} = i_0[n]^{m-1}$, we have

$$C_{\mathbf{i}}^k \subset \Psi_{i_0[n]^{m-1}}(C^1), \quad R_{\ell}(T_{i_0}) = \Psi_{i_0[n]^{m-1}} \left(\bigcup_{j=n-\beta+1}^n T_{[n]^{\ell-m+1}j} \right).$$

Notice that

$$\begin{aligned} \max C^1 &= \Psi_{(i_0+1)[1]^{\tau(1)}\alpha}(1) \leq \Psi_{n1\alpha}(1) < \Psi_{n1}(1), \\ \min T_{[n]^{\ell-m+1}(n-\beta+1)} &= \Psi_{[n]^{\ell-m+1}(n-\beta+1)}(0) > \Psi_{n(n-\beta+1)}(0) > \Psi_{n(n-\beta)}(1) > \max C^1. \end{aligned}$$

We have $C_{\mathbf{i}}^k \cap C^{\ell} = C_{\mathbf{i}}^k \cap R_{\ell}(T_{i_0}) = \emptyset$.

In case that $\mathbf{i} = (i_0 + 1)[1]^{m-1}$, we have

$$C_{\mathbf{i}}^k \subset \Psi_{(i_0+1)[1]^{m-1}}(C^1), \quad L_{\tau(\ell)}(T_{i_0+1}) = \Psi_{(i_0+1)[1]^{m-1}} \left(\bigcup_{j=1}^{\alpha} T_{[1]^{\tau(\ell)-m+1}j} \right).$$

Notice that

$$\begin{aligned} \min C^1 &= \Psi_{i_0 n(n-\beta+1)}(0) > \Psi_{1n(n-\beta)}(1) > \Psi_{1(n-1)}(1), \\ \max T_{[1]^{\tau(\ell)-m+1}\alpha} &< \Psi_{1\alpha}(1) \leq \Psi_{1(n-1)}(1) < \min C^1, \end{aligned}$$

where we use $\tau(\ell) - m \geq \ell - m = k \geq 1$. We have $C_{\mathbf{i}}^k \cap C^{\ell} = C_{\mathbf{i}}^k \cap L_{\tau(\ell)}(T_{i_0+1}) = \emptyset$.

Case 2. Assume that $\mathbf{j} \neq \emptyset$. Let $\mathbf{u} \in \Sigma_n^* \cup \{\emptyset\}$ be the word with the maximal length which satisfies $\mathbf{i} = \mathbf{u}\mathbf{i}'$ and $\mathbf{j} = \mathbf{u}\mathbf{j}'$ for some $\mathbf{i}', \mathbf{j}' \in \Sigma_n^* \cup \{\emptyset\}$.

Suppose that $\mathbf{u} \neq \mathbf{j}$, then \mathbf{i}', \mathbf{j}' are all in Σ_n^* with $\mathbf{i}'(1) \neq \mathbf{j}'(1)$, where $\mathbf{i}'(1)$ and $\mathbf{j}'(1)$ is the first letter of \mathbf{i}' and \mathbf{j}' , respectively. Using $C_{\mathbf{i}'}^k \subset T_{\mathbf{i}'(1)}$ and $C_{\mathbf{j}'}^{\ell} \subset T_{\mathbf{j}'(1)}$, it is easy to see that $C_{\mathbf{i}'}^k \cap C_{\mathbf{j}'}^{\ell} = \emptyset$ so that $C_{\mathbf{i}}^k \cap C_{\mathbf{j}}^{\ell} = \Psi_{\mathbf{u}}(C_{\mathbf{i}'}^k \cap C_{\mathbf{j}'}^{\ell}) = \emptyset$.

Suppose that $\mathbf{u} = \mathbf{j}$. Using the result of Case 1, we have $C_{\mathbf{i}}^k \cap C_{\mathbf{j}}^{\ell} = \Psi_{\mathbf{j}}(C_{\mathbf{i}'}^k \cap C^{\ell}) = \emptyset$. ■

Lemma 2.4. *For any $A \in \mathcal{C}_u$ and $B \in \mathcal{C}_v$ with $u > v$. We have either $A \cap B = \emptyset$ or $A \subset B$.*

Proof. Suppose that $A = C_{\mathbf{i}}^k$ and $B = C_{\mathbf{j}}^{\ell}$.

If $\mathbf{i} = \mathbf{j} = \emptyset$, then the lemma holds in this case since $C^k \subset C^{\ell}$ for $k > \ell$. If $\mathbf{i} = \emptyset$ and $\mathbf{j} \in \Sigma_n^*$, then from Lemma 2.3, we have $C^k \cap C_{\mathbf{j}}^{\ell} \subset C^{|\mathbf{j}|+\ell} \cap C_{\mathbf{j}}^{\ell} = \emptyset$ so that the lemma also holds in this case. Thus, we can assume that $\mathbf{i} \in \Sigma_n^*$ in the following.

Given $\mathbf{i} \in \Sigma_n^*$. It is easy to check that we must have either $C_{\mathbf{i}}^1 \cap R_{\ell}(T_{i_0}) = \emptyset$ or $C_{\mathbf{i}}^1 \subset R_{\ell}(T_{i_0})$, while $C_{\mathbf{i}}^1 \subset R_{\ell}(T_{i_0})$ if and only if one of the followings happens: (1). $\mathbf{i} = i_0[n]^{\ell}$ and $i_0 \geq n - \beta + 1$, or (2). $\mathbf{i} = i_0[n]^{\ell}j\mathbf{u}$ for some $j \in \{n - \beta + 1, \dots, n\}$ and $\mathbf{u} \in \Sigma_n^* \cup \{\emptyset\}$. Similarly, we must have either $C_{\mathbf{i}}^1 \cap L_{\tau(\ell)}(T_{i_0+1}) = \emptyset$ or $C_{\mathbf{i}}^1 \subset L_{\tau(\ell)}(T_{i_0+1})$, while $C_{\mathbf{i}}^1 \subset L_{\tau(\ell)}(T_{i_0+1})$ if and only if one of the followings

happens: (1). $\mathbf{i} = (i_0 + 1)[1]^{\tau(\ell)}$ and $i_0 \leq \alpha - 1$, or (2). $\mathbf{i} = (i_0 + 1)[1]^{\tau(\ell)}j\mathbf{u}$ for some $j \in \{1, \dots, \alpha\}$ and $\mathbf{u} \in \Sigma_n^* \cup \{\emptyset\}$. It follows that we must have either $C_{\mathbf{i}}^1 \cap C^\ell = \emptyset$ or $C_{\mathbf{i}}^1 \subset C^\ell$.

Case 1. Assume that $\mathbf{j} = \emptyset$. Since $C_{\mathbf{i}}^k \subset C_{\mathbf{i}}^1$ for any $k \in \mathbb{Z}^+$, we know from above that the lemma holds in this case.

Case 2. Assume that $\mathbf{j} \neq \emptyset$. Let $\mathbf{u} \in \Sigma_n^* \cup \{\emptyset\}$ be the word with the maximal length which satisfies $\mathbf{i} = \mathbf{u}\mathbf{i}'$ and $\mathbf{j} = \mathbf{u}\mathbf{j}'$ for some $\mathbf{i}', \mathbf{j}' \in \Sigma_n^* \cup \{\emptyset\}$.

Suppose that both \mathbf{i}' and \mathbf{j}' are in Σ_n^* . Using the same method in Case 2 in the proof of Lemma 2.3, we have $C_{\mathbf{i}}^k \cap C_{\mathbf{j}}^\ell = \Psi_{\mathbf{u}}(C_{\mathbf{i}'}^k \cap C_{\mathbf{j}'}^\ell) = \emptyset$.

Suppose that one of \mathbf{i}' and \mathbf{j}' equals \emptyset . Using the above discussions, we can easily see that one of the followings must holds: $C_{\mathbf{i}}^k \cap C_{\mathbf{j}}^\ell = \emptyset$ or $C_{\mathbf{i}}^k \subset C_{\mathbf{j}}^k$. ■

Let E be a given subset of \mathbb{R} and \mathcal{P} a family of finitely many closed subsets of E . If $\bigcup_{A \in \mathcal{P}} A = E$ and the union is disjoint, we call \mathcal{P} a *partition* of E and define $\|\mathcal{P}\| = \max_{A \in \mathcal{P}} \text{diam } A$. Let \mathcal{A}_1 and \mathcal{A}_2 be two partitions of E . If for any $A \in \mathcal{A}_1$, there exist $j \in \mathbb{Z}^+$ and $A'_1, \dots, A'_j \in \mathcal{A}_2$ such that $A = \bigcup_{i=1}^j A'_i$, then \mathcal{A}_2 is called a *refinement* of \mathcal{A}_1 . Clearly, \mathcal{A}_2 is a refinement of \mathcal{A}_1 if and only if for each $B \in \mathcal{A}_2$, there exists $B' \in \mathcal{A}_1$ such that $B \subset B'$.

Let $\{\mathcal{P}_k\}$ be a sequence of partitions of T . $\{\mathcal{P}_k\}$ is called *hierarchical* if \mathcal{P}_{k+1} is a refinement of \mathcal{P}_k for any k . $\{\mathcal{P}_k\}$ is called *convergent* if it is hierarchical and $\lim_{k \rightarrow \infty} \|\mathcal{P}_k\| = 0$.

Denote by $\text{card } A$ the cardinality of A for any set A . Given a bounded subset B of \mathbb{R} . We define $CH(B)$ to be the convex hull of B . Equivalently, $CH(B)$ is the minimal closed interval containing B .

Let A be a given compact subset of \mathbb{R} . Let $\{A_i\}_{i=1}^k$ be a family of compact subsets of A with the following properties: A_i is A -separate for all i , $CH(A_i) \cap A = A_i$ for all i , and $CH(A_i)$ does not intersect $CH(A_j)$ for all distinct i and j . We define \mathcal{S} to be the family of compact subsets of A with the minimal cardinality such that the following two conditions hold: (1). $\bigcup_{B \in \mathcal{S}} B = A$ and $CH(B) \cap CH(B') = \emptyset$ for all distinct $B, B' \in \mathcal{S}$; (2). $A_i \in \mathcal{S}$ for all i . We call \mathcal{S} the *simple decomposition* of A by $\{A_i\}_{i=1}^k$. Clearly, there exists a unique simple decomposition for given A and $\{A_i\}_{i=1}^k$. Furthermore, it is obvious that we have the following property by definition:

$$(2.7) \quad \text{If } B \subset A \text{ and } CH(B) \cap \bigcup_{i=1}^k A_i = \emptyset, \text{ then there exists a unique } E \in \mathcal{S}, \text{ such that } B \subset E.$$

We call this the *containing property* of the simple decomposition. For convenience, $\mathcal{S} = \{A\}$ is defined to be the simple decomposition of A by \emptyset . It is clear that the containing property still holds in this case.

Given $k \in \mathbb{Z}^+$ and a compact subset F of T , we define $\mathcal{C}_k(F) = \{A : A \in \mathcal{C}_k \text{ and } A \subset F\}$. Notice that A is F -separate and $CH(A) \cap F = A$ for all $A \in \mathcal{C}_k(F)$. We define $\mathcal{S}_k(F)$ to be the simple decomposition of F by $\mathcal{C}_k(F)$.

Now we inductively construct $\{\mathcal{S}_k\}$ as follows. Define $\mathcal{S}_1 = \mathcal{S}_1(T)$ and $\mathcal{S}_{k+1} = \bigcup_{F \in \mathcal{S}_k} \mathcal{S}_{k+1}(F)$ for $k \geq 1$. Clearly, \mathcal{S}_1 is the simple decomposition of T by $\{C^1\}$, i.e. $\mathcal{S}_1 = \{[0, a] \cap T, C^1, [b, 1] \cap T\}$, where $a = \Psi_{i_0 n(n-\beta)}(1)$ and $b = \Psi_{(i_0+1)[1]^{\tau(1)}(\alpha+1)}(0)$.

Lemma 2.5. *$\{\mathcal{S}_k\}$ is a hierarchical partition sequence of T such that $\mathcal{C}_k \subset \mathcal{S}_k$ for all positive integers k .*

Proof. By the definition of the simple decomposition, we know that \mathcal{S}_{k+1} is a refinement of \mathcal{S}_k for all k , and \mathcal{S}_k is a partition of T for all k . Thus, in order to prove the lemma, it suffices to show that $\mathcal{C}_k \subset \mathcal{S}_k$ for all k . We will prove this by induction. Clearly, $\mathcal{C}_1 \subset \mathcal{S}_1$.

Assume that $\mathcal{C}_k \subset \mathcal{S}_k$ for all $k \leq m$ for some given $m \in \mathbb{Z}^+$.

Claim 1. *For any $A \in \mathcal{C}_{m+1}$, there exists $B \in \mathcal{S}_m$ such that $A \subset B$.*

Proof. Since $A \subset T$ and \mathcal{S}_m is a partition of T , there exists $B \in \mathcal{S}_m$ such that $A \cap B \neq \emptyset$. Denote this B by B_m . Notice that by definition, \mathcal{S}_{k+1} is a refinement of \mathcal{S}_k for all k . Hence there exists (unique) $\{B_k\}_{k=1}^{m-1}$ such that $B_{k+1} \subset B_k$ and $B_k \in \mathcal{S}_k$ for all $k = 1, 2, \dots, m-1$. It follows from $A \cap B_m \neq \emptyset$ that $A \cap B_k \neq \emptyset$ for all $k = 1, 2, \dots, m$.

Suppose that $A \cap C^1 \neq \emptyset$. Then by Lemma 2.4, we have $A \subset C^1$. Since all sets in the family \mathcal{S}_1 are disjoint, we have $B_1 = C^1$ so that $A \subset B_1$. Suppose that $A \cap C^1 = \emptyset$. Notice that $A \subset T$ and \mathcal{S}_1 is the simple decomposition of T by $\{C^1\}$. Using the containing property of the simple decomposition, there exists a unique $E \in \mathcal{S}_1 \setminus \{C^1\}$ such that $A \subset E$. By the same reason as above, we have $B_1 = E$ so that $A \subset B_1$. Hence, we always have $A \subset B_1$. Thus the claim holds in case that $m = 1$.

Assume that $m \geq 2$. Suppose that there exists $B' \in \mathcal{C}_2$ such that $A \cap B' \neq \emptyset$. Similarly as above, we have $A \subset B'$. By the inductive assumption, we have $\mathcal{C}_2 \subset \mathcal{S}_2$. Since \mathcal{S}_2 is a partition of T , we have $B_2 = B'$ so that $A \subset B_2$. Suppose that $A \cap \bigcup_{F \in \mathcal{C}_2} F = \emptyset$. Then we have $A \cap \bigcup_{F \in \mathcal{C}_2(B_1)} F = \emptyset$. Since $A \subset B_1$, we can obtain from the containing property of the simple decomposition that there exists $B' \in \mathcal{S}_2(B_1) \setminus \mathcal{C}_2(B_1) \subset \mathcal{S}_2$ such that $A \subset B'$. Similarly as above, we have $B_2 = B'$ so that $A \subset B_2$. Repeating this process, we can see that $A \subset B_k$ for $k = 1, 2, \dots, m$. Thus the claim also holds in this case. ■

From the above claim, we know that for each $A \in \mathcal{C}_{m+1}$, there exists $B \in \mathcal{S}_m$ such that $A \subset B$. Thus, we have $A \in \mathcal{S}_{m+1}(B) \subset \mathcal{S}_{m+1}$. It follows that $\mathcal{C}_k \subset \mathcal{S}_k$ for $k = m+1$. By induction, $\mathcal{C}_k \subset \mathcal{S}_k$ for all $k \in \mathbb{Z}^+$. ■

By Lemma 2.5, we can show that the following corollary holds.

Corollary 2.1. *There exists a convergent partition sequence $\{\mathcal{T}_k\}$ of T such that $\mathcal{C}_k \subset \mathcal{T}_k$ for all positive integers k .*

Proof. Let E be a compact subset of T . An open interval (a, b) is said to be a *gap* of E if $a, b \in E$ and $(a, b) \cap E = \emptyset$. We call $b - a$ the *length* of the gap (a, b) . Let δ be a positive real number. We define $\mathcal{G}(E, \delta) = \{(a, b) : (a, b) \text{ is a gap of } E \text{ such that } b - a \geq \delta\}$. Define $\mathcal{J}(E, \delta)$ to be the family of all connected components of $CH(E) \setminus \bigcup_{F \in \mathcal{G}(E, \delta)} F$. Define $\mathcal{P}(E, \delta) = \{A \cap E : A \in \mathcal{J}(E, \delta)\}$. Then $\mathcal{P}(E, \delta)$ is a partition of E . Furthermore, for all $F \in \mathcal{P}(E, \delta)$, $CH(F)$ does not contain any gap of T whose length greater than δ .

Now, we define $\delta_k = \max\{\text{diam } A : A \in \mathcal{C}_k\}$ for $k \in \mathbb{Z}^+$. Then the sequence $\{\delta_k\}_{k=1}^\infty$ is decreasing and $\lim_{k \rightarrow \infty} \delta_k = 0$. Define

$$\mathcal{T}_k = \bigcup_{E \in \mathcal{S}_k} \mathcal{P}(E, \delta_k), \quad \forall k.$$

Clearly, \mathcal{T}_k is also a hierarchical partition sequence of T with $\mathcal{C}_k \subset \mathcal{T}_k$ for all k . From $\mathcal{C}_1^1 \subset I_1$, we can see that for any $A \in \mathcal{I}_k$ with $k \in \mathbb{Z}^+$, there exists $B \in \mathcal{C}_{k+1}$ such that $B \subset A$. Thus

$$\|\mathcal{T}_{k+1}\| < 2 \cdot \max\{\text{diam } A : A \in \mathcal{I}_k\} + \delta_{k+1}$$

for all k so that $\lim_{k \rightarrow \infty} \|\mathcal{T}_k\| = 0$. It follows that the corollary holds. ■

2.3. Martingales and the proof of Theorem 1.1. Assume that $f : T \rightarrow D$ is bi-Lipschitz, i.e., f is bijective and there exist two positive constants c, c' such that

$$(2.8) \quad c|x - y| \leq |f(x) - f(y)| \leq c'|x - y|, \quad \forall x, y \in T.$$

Let s be the common Hausdorff dimension of T and D , i.e. $\dim_H T = \dim_H D = s$.

By Lemmas 2.1 and 2.2, there exists an integer n_0 such that for any $\mathbf{i} \in \Sigma_n^* \cup \{\emptyset\}$ and $k \in \mathbb{Z}^+$, there exist $\mathbf{j}, \mathbf{j}_1, \dots, \mathbf{j}_p \in \Sigma_n^*$ such that $D_{\mathbf{j}\mathbf{j}_1}, D_{\mathbf{j}\mathbf{j}_2}, \dots, D_{\mathbf{j}\mathbf{j}_p}$ are disjoint and

$$f(C_{\mathbf{i}}^k) = \bigcup_{r=1}^p D_{\mathbf{j}\mathbf{j}_r} \subset D_{\mathbf{j}},$$

where each $|\mathbf{j}_r| = n_0$. Furthermore, by Remark 2.1, we can require $D_{\mathbf{j}}$ to be the smallest cylinder containing $f(C_{\mathbf{i}}^k)$. We denote this \mathbf{j} by $\mathbf{j}(\mathbf{i}, k)$ and define $\gamma_{\mathbf{i}, k} = \sum_{r=1}^p \rho_{\mathbf{j}_r}^s$. Then

$$(2.9) \quad \mathcal{H}^s(f(C_{\mathbf{i}}^k)) = \mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}, k)}) \cdot \gamma_{\mathbf{i}, k}, \quad \text{and}$$

$$(2.10) \quad D_{\mathbf{j}(\mathbf{i}, k)} \subset D_{\mathbf{j}(\mathbf{i}', k')} \quad \text{if} \quad C_{\mathbf{i}}^k \subset C_{\mathbf{i}'}^{k'}.$$

Define

$$(2.11) \quad \mathcal{M} = \left\{ \sum_{\mathbf{j} \in \mathcal{A}} \rho_{\mathbf{j}}^s \mid \mathcal{A} \subset \{1, \dots, n\}^{n_0} \right\}.$$

Then $\gamma_{\mathbf{i}, k} \in \mathcal{M}$ for all \mathbf{i} and k .

Let $\{\mathcal{T}_k\}$ be a convergent partition sequence of T as defined in Corollary 2.1. We define

$$g_k(A) = \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)}, \quad A \in \mathcal{T}_k.$$

Notice that for any $A \in \mathcal{T}_k$, we can decompose A by $A = \bigcup_{i=1}^j A'_i$ where $A'_1, \dots, A'_j \in \mathcal{T}_{k+1}$. Then,

$$(2.12) \quad \sum_{i=1}^j \frac{\mathcal{H}^s(A'_i)}{\mathcal{H}^s(A)} g_{k+1}(A'_i) = \sum_{i=1}^j \frac{\mathcal{H}^s(A'_i)}{\mathcal{H}^s(A)} \frac{\mathcal{H}^s(f(A'_i))}{\mathcal{H}^s(A'_i)} = \sum_{i=1}^j \frac{\mathcal{H}^s(f(A'_i))}{\mathcal{H}^s(A)} = \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)} = g_k(A).$$

Let \mathcal{F}_k be the sigma field generated by \mathcal{T}_k , and define

$$g_k(x) = g_k(A)$$

for $x \in A$ where $A \in \mathcal{T}_k$. Then g_k is a \mathcal{F}_k -measurable function. By (2.12), we know that (g_k, \mathcal{F}_k) is a martingale. Furthermore, by (2.8), we have $c^s \leq g_k(x) \leq (c')^s$ for any k and any $x \in T$. So the martingale convergence theorem implies that

$$(2.13) \quad g_k(x) \rightarrow g(x) \quad \text{as } k \rightarrow \infty, \quad \text{for } \mathcal{H}^s\text{-almost all } x \text{ in } T,$$

where g is \mathcal{F} -measurable, with \mathcal{F} the sigma field generated by $\bigcup_{k=1}^{\infty} \mathcal{F}_k$.

Define $\mu_i = \rho_i^s$ for any $i = 1, \dots, n$. For any $\mathbf{i} = i_1 \cdots i_j \in \Sigma_n^*$, we define $\mu_{\mathbf{i}} = \prod_{k=1}^j \mu_{i_k}$. Denote $\mu_L = \sum_{j=1}^{\alpha} \mu_j$ and $\mu_R = \sum_{j=n-\beta+1}^n \mu_j$. We have the following lemma.

Lemma 2.6. *If $D \sim T$, then there exist a constant $M > 0$ and infinitely many $(p_1, p_2, q_1, q_2) \in (\mathbb{Z}^+)^4$ with $q_1 \neq q_2$ and $|p_2 - p_1|, |q_2 - q_1| \leq M$ such that*

$$(2.14) \quad \frac{\mu_{i_0} \mu_n^{q_2} \mu_R + \mu_{i_0+1} \mu_1^{p_2} \mu_L}{\mu_{i_0} \mu_n^{q_1} \mu_R + \mu_{i_0+1} \mu_1^{p_1} \mu_L} = \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n},$$

where $\{k_i\}_{i=1}^n$ are nonnegative integers with $\max_i k_i \leq M$.

Proof. Let $\Sigma_n^\infty = \{i_1 i_2 \cdots i_m \cdots \mid i_m \in \{1, \dots, n\} \text{ for all } m\}$. For each $x \in T$, there exists $i_1 \cdots i_m \cdots \in \Sigma_n^\infty$ such that $\{x\} = \bigcap_{m \geq 1} \Psi_{i_1 \cdots i_m}(T)$. We call $i_1 \cdots i_m \cdots$ to be the address of x . We remark that the address of x may be not unique. However, if we define \tilde{T} to be the set of all points in T with unique address, then $\mathcal{H}^s(\tilde{T}) = \mathcal{H}^s(T)$ by the definition of T . For each $x \in \tilde{T}$ with address $i_1 \cdots i_m \cdots$, we define $\sigma(x)$ to be the point with address $i_2 \cdots i_m \cdots$. It is easy to check that $\sigma(x) \in \tilde{T}$ for all $x \in \tilde{T}$.

Let $\tilde{\mathcal{F}}$ be the sigma field generated by $\{T_1 \cap \tilde{T} : \mathbf{i} \in \Sigma_n^*\}$. Define $\nu(A) = \mathcal{H}^s(A \cap \tilde{T}) / \mathcal{H}^s(\tilde{T})$, $\forall A \in \tilde{\mathcal{F}}$. Then $\sigma : (\tilde{T}, \tilde{\mathcal{F}}, \nu) \rightarrow (\tilde{T}, \tilde{\mathcal{F}}, \nu)$ is measure preserving. Fix $p \geq \text{card } \mathcal{M} + 1$ in the proof of the lemma, where \mathcal{M} is defined by (2.11). Given $q \in \mathbb{Z}^+$, by the Poincaré recurrence theorem, for ν -almost all $x \in C^{pq} \cap \tilde{T}$, i.e. for \mathcal{H}^s -almost all $x \in C^{pq} \cap \tilde{T}$, there is an integer sequence $0 < n_1(x, q) < n_2(x, q) < \cdots$ such that $\sigma^{n_i(x, q)}(x) \in C^{pq} \cap \tilde{T}$ for all i . Thus, from (2.13), we can pick a point $x_q \in C^{pq} \cap \tilde{T}$ with $\sigma^{n_i(x_q, q)}(x_q) \in C^{pq} \cap \tilde{T}$ for each i and $g_k(x_q) \rightarrow g(x_q)$ as $k \rightarrow \infty$.

Let $i_1 \cdots i_m \cdots$ be the address of x_q . Define $\mathbf{i}_k = i_1 i_2 \cdots i_{n_k(x_q, q)}$. Then

$$x_q = \Psi_{\mathbf{i}_k}(\sigma^{n_k(x_q, q)}(x_q)) \in \Psi_{\mathbf{i}_k}(C^{pq}) = C_{\mathbf{i}_k}^{pq}.$$

Recall that $\mathcal{H}^s(f(C_{\mathbf{i}_k}^t)) = \mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, t)}) \cdot \gamma_{\mathbf{i}_k, t}$ for any $t \in \mathbb{Z}^+$. For any $t, t' \in \mathbb{Z}^+ \cap [1, p]$ with $t' < t$, using $x_q \in C_{\mathbf{i}_k}^{pq} \subset C_{\mathbf{i}_k}^{p(q-1)+t} \subset C_{\mathbf{i}_k}^{p(q-1)+t'}$, we have

$$\frac{g_{|\mathbf{i}_k|+p(q-1)+t}(x_q)}{g_{|\mathbf{i}_k|+p(q-1)+t'}(x_q)} = \frac{\mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t)})}{\mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t')})} \cdot \frac{\gamma_{\mathbf{i}_k, p(q-1)+t}}{\gamma_{\mathbf{i}_k, p(q-1)+t'}} \cdot \frac{\mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)+t'})}{\mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)+t})}.$$

By (2.10), $D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t)} \subset D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t')}$ so that there exists $\mathbf{u}(q, k, t, t') \in \Sigma_n^*$ such that

$$\mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t)}) = \mu_{\mathbf{u}(q, k, t, t')} \mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t')}).$$

Meanwhile, by definition,

$$\frac{\mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)+t'})}{\mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)+t})} = \frac{\mu_{i_0} \mu_n^{p(q-1)+t'} \mu_R + \mu_{i_0+1} \mu_1^{\tau(p(q-1)+t')} \mu_L}{\mu_{i_0} \mu_n^{p(q-1)+t} \mu_R + \mu_{i_0+1} \mu_1^{\tau(p(q-1)+t)} \mu_L}.$$

Thus

$$(2.15) \quad \frac{g_{|\mathbf{i}_k|+p(q-1)+t}(x_q)}{g_{|\mathbf{i}_k|+p(q-1)+t'}(x_q)} = \mu_{\mathbf{u}(q, k, t, t')} \cdot \frac{\gamma_{\mathbf{i}_k, p(q-1)+t}}{\gamma_{\mathbf{i}_k, p(q-1)+t'}} \cdot \frac{\mu_{i_0} \mu_n^{p(q-1)+t'} \mu_R + \mu_{i_0+1} \mu_1^{\tau(p(q-1)+t')} \mu_L}{\mu_{i_0} \mu_n^{p(q-1)+t} \mu_R + \mu_{i_0+1} \mu_1^{\tau(p(q-1)+t)} \mu_L}.$$

Claim 2. $\{\mu_{\mathbf{u}(q, k, t, t')}\}_{q \geq 1, k \geq 1, 1 \leq t' < t \leq p}$ can take only finitely many values.

Proof. Notice that f is bi-Lipschitz. Thus for all $q \geq 1, k \geq 1, 1 \leq t' < t \leq p$ we have

$$\mathcal{H}^s(f(C_{\mathbf{i}_k}^{p(q-1)+t})) \asymp \mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)+t}) \asymp \mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)}) \asymp \mathcal{H}^s(C_{\mathbf{i}_k}^{p(q-1)+t'}) \asymp \mathcal{H}^s(f(C_{\mathbf{i}_k}^{p(q-1)+t'})).$$

On the other hand, by (2.9),

$$\mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t)}) \asymp \mathcal{H}^s(f(C_{\mathbf{i}_k}^{p(q-1)+t})) \quad \text{and} \quad \mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t')}) \asymp \mathcal{H}^s(f(C_{\mathbf{i}_k}^{p(q-1)+t'}))$$

for all $q \geq 1, k \geq 1, 1 \leq t' < t \leq p$. Thus

$$\mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t)}) \asymp \mathcal{H}^s(D_{\mathbf{j}(\mathbf{i}_k, p(q-1)+t')})$$

so that $\mu_{\mathbf{u}(q, k, t, t')} \asymp 1$. The claim follows immediately. ■

By the claim, for fixed $q \geq 1$, the right hand of (2.15) can take only finitely many distinct values for $1 \leq t' < t \leq p$ and $k \in \mathbb{Z}^+$ so that $\frac{g_{|\mathbf{i}_k|+p(q-1)+t}(x_q)}{g_{|\mathbf{i}_k|+p(q-1)+t'}(x_q)}$ can take only finitely many distinct values. Hence, we can take k large enough such that $\frac{g_{|\mathbf{i}_k|+p(q-1)+t}(x_q)}{g_{|\mathbf{i}_k|+p(q-1)+t'}(x_q)}$ is so close to 1 that it equals 1. Since $p \geq \text{card } \mathcal{M} + 1$, we can take $t_q > t'_q$ such that $\gamma_{\mathbf{i}_k, p(q-1)+t_q} = \gamma_{\mathbf{i}_k, p(q-1)+t'_q}$. Thus,

$$\frac{\mu_{i_0} \mu_n^{p(q-1)+t_q} \mu_R + \mu_{i_0+1} \mu_1^{\tau(p(q-1)+t_q)} \mu_L}{\mu_{i_0} \mu_n^{p(q-1)+t'_q} \mu_R + \mu_{i_0+1} \mu_1^{\tau(p(q-1)+t'_q)} \mu_L} = \mu_1^{k_1(q)} \mu_2^{k_2(q)} \cdots \mu_n^{k_n(q)},$$

where $\{k_i(q)\}_{1 \leq i \leq n, q \geq 1}$ are bounded nonnegative integers. Also, we have $|(p(q-1)+t_q)-(p(q-1)+t'_q)| \leq p-1$. From

$$\rho_1^{\tau(p(q-1)+t_q)} \asymp \rho_n^{p(q-1)+t_q} \asymp \rho_n^{p(q-1)+t'_q} \asymp \rho_1^{\tau(p(q-1)+t'_q)}$$

with respect to the indices p, q we know that $\{\tau(p(q-1)+t_q) - \tau(p(q-1)+t'_q)\}_{q \geq 1}$ are bounded.

Define $p_1 = \tau(p(q-1)+t'_q)$, $q_1 = p(q-1)+t'_q$, $p_2 = \tau(p(q-1)+t_q)$, $q_2 = p(q-1)+t_q$. From $1 \leq t'_q < t_q \leq p$, we have $q_1, q_2 \in \mathbb{Z}^+ \cap [p(q-1)+1, pq]$ with $q_1 \neq q_2$. Since q can be arbitrary chosen in \mathbb{Z}^+ , we finally obtain infinitely many solution of (2.14). The lemma is proved. \blacksquare

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.6, there exist infinitely many solutions (p_1, p_2, q_1, q_2) of (2.14) with $q_1 \neq q_2$ such that $(p_2 - p_1), (q_2 - q_1), \{k_i\}_{i=1}^n$ are constants. It follows that there are infinitely many $(p_1, q_1) \in (\mathbb{Z}^+)^2$ and constants $\mu_{i_0} \mu_n^{q_2 - q_1} \mu_R$, $\mu_{i_0+1} \mu_1^{p_2 - p_1} \mu_L$, $\mu_{i_0} \mu_R$, $\mu_{i_0+1} \mu_L$, $\mu_1^{k_1} \cdots \mu_n^{k_n}$ such that the following equation holds:

$$\frac{(\mu_{i_0} \mu_n^{q_2 - q_1} \mu_R) \cdot \mu_n^{q_1} + (\mu_{i_0+1} \mu_1^{p_2 - p_1} \mu_L) \cdot \mu_1^{p_1}}{(\mu_{i_0} \mu_R) \cdot \mu_n^{q_1} + (\mu_{i_0+1} \mu_L) \cdot \mu_1^{p_1}} = \mu_1^{k_1} \mu_2^{k_2} \cdots \mu_n^{k_n}.$$

Assume that $\frac{\mu_{i_0} \mu_n^{q_2 - q_1} \mu_R}{\mu_{i_0} \mu_R} \neq \frac{\mu_{i_0+1} \mu_1^{p_2 - p_1} \mu_L}{\mu_{i_0+1} \mu_L}$. Then there is a constant δ such that $\frac{\mu_1^{p_1}}{\mu_n^{q_1}} = \delta$ for infinitely many $(p_1, q_1) \in (\mathbb{Z}^+)^2$. Take $(p_1, q_1), (p'_1, q'_1) \in (\mathbb{Z}^+)^2$ such that $(p_1, q_1) \neq (p'_1, q'_1)$ and $\frac{\mu_1^{p_1}}{\mu_n^{q_1}} = \delta = \frac{\mu_1^{p'_1}}{\mu_n^{q'_1}}$. It follows that $\mu_1^{p_1 - p'_1} = \mu_n^{q_1 - q'_1}$ so that $\rho_1^{p_1 - p'_1} = \rho_n^{q_1 - q'_1}$ which implies that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$.

Assume that $\frac{\mu_{i_0} \mu_n^{q_2 - q_1} \mu_R}{\mu_{i_0} \mu_R} = \frac{\mu_{i_0+1} \mu_1^{p_2 - p_1} \mu_L}{\mu_{i_0+1} \mu_L}$. Then $\mu_n^{q_2 - q_1} = \mu_1^{p_2 - p_1}$ with $q_2 \neq q_1$ so that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$. \blacksquare

2.4. Algebraic dependence necessary condition for $n = 4$. In this subsection, we assume that $n = 4$, $\rho_1 = \rho_4$, $\Sigma_T = \{2\}$ and $f : T \rightarrow D$ is bi-Lipschitz. Denote $s := \dim_H T = \dim_H D$. Denote $\mu_i := \rho_i^s$ for each i .

Define $E_k = T_{2[4]^k} \cup T_{3[1]^k}$ for all $k \geq 0$. Define $\mathcal{E}_1 = \{T_1, T_4, E_0\}$ and

$$\mathcal{E}_{k+1} = \Psi_1(\mathcal{E}_k) \cup \Psi_4(\mathcal{E}_k) \cup (\Psi_2(\mathcal{E}_k) \setminus \{T_{2[4]^k}\}) \cup (\Psi_3(\mathcal{E}_k) \setminus \{T_{3[1]^k}\}) \cup \{E_k\}$$

for all $k \geq 1$. For example, the sets in the class \mathcal{E}_2 are:

$$T_{11}, T_{14}, \Psi_1(E_0), T_{41}, T_{44}, \Psi_4(E_0), T_{21}, \Psi_2(E_0), T_{34}, \Psi_3(E_0), E_1.$$

Remark 2.2. Let $i_0 = 2$ and $\{C^k\}_{k=1}^\infty$ defined as in subsection 2.2. Then $C^k = T_{2[4]^{k+1}} \cup T_{3[1]^{k+1}} = E_{k+1}$ for $k \geq 1$. Also, it is clear that the sequence $\{\mathcal{E}_k\}_{k=1}^\infty$ is a convergent partition of T . However, in this subsection, we will not use these facts and the martingale convergent theorem.

It is clear that the following lemma holds.

Lemma 2.7. *Let $\lambda_0 = \min\{\frac{\text{diam}(A)}{d(A, T \setminus A)} : A \in \mathcal{E}_1\}$. Then each set in the family $\bigcup_{k=1}^{\infty} \mathcal{E}_k$ is (T, λ_0) -separate.*

For any $A \in \mathcal{E}_k$, we define

$$\tilde{g}_k(A) = \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(A)}.$$

We also abuse the notation $\tilde{g}_k(x) = \tilde{g}_k(A)$ for $x \in A$. Assume that $\{A_1, \dots, A_j\}$ be a partition of A in \mathcal{E}_{k+1} , i.e. $A = \bigcup_{i=1}^j A_i$, $A_i \in \mathcal{E}_{k+1}$ for each i , and the union is disjoint. Then it is clear that

$$\tilde{g}_k(A) = \sum_{i=1}^j \frac{\mathcal{H}^s(A_i)}{\mathcal{H}^s(A)} \tilde{g}_{k+1}(A_i).$$

Lemma 2.8. *The set $\{\frac{\tilde{g}_{k+1}(x)}{\tilde{g}_k(x)} : x \in T, k \geq 1\}$ is finite.*

Proof. Notice that $\frac{\mathcal{H}^s(E_{k+1})}{\mathcal{H}^s(E_k)} = \mu_1$ for all k . By induction, we can easily see that

$$\left\{ \frac{\mathcal{H}^s(A)}{\mathcal{H}^s(B)} : A \in \mathcal{E}_{k+1}, B \in \mathcal{E}_k \text{ with } A \subset B \right\} = \left\{ \mu_1, \mu_2, \mu_3, \mu_2 + \mu_3, \frac{\mu_1 \mu_2}{\mu_2 + \mu_3}, \frac{\mu_1 \mu_3}{\mu_2 + \mu_3} \right\}$$

for all $k \geq 1$. On the other hand, using Lemma 2.1, Remark 2.1 and Lemma 2.7, and using the bi-Lipschitz property of f , we can obtain that the set

$$\left\{ \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(f(B))} : A \in \mathcal{E}_{k+1}, B \in \mathcal{E}_k \text{ with } A \subset B, k \geq 1 \right\}$$

is finite.

Given $x \in T$ and $k \geq 1$. We assume that $x \in A \subset B$ with $A \in \mathcal{E}_{k+1}$ and $B \in \mathcal{E}_k$. Then

$$\frac{\tilde{g}_{k+1}(x)}{\tilde{g}_k(x)} = \frac{\mathcal{H}^s(f(A))}{\mathcal{H}^s(f(B))} \cdot \frac{\mathcal{H}^s(B)}{\mathcal{H}^s(A)}.$$

By above discussions, we know that the lemma holds. ■

Now we have the following property by using Lemma 2.8. We remark that the proof of this property is same as the proof of Lemma 4 in [21], which was restated in [12] for completeness (see the proof of Lemma 2.4 therein). Thus we omit the proof.

Lemma 2.9. *There is a set A_0 in the family $\bigcup_{k=1}^{\infty} \mathcal{E}_k$ and a constant $\delta > 0$, such that $\tilde{g}_k(x) = \delta$ for all $x \in A_0$ and $k \geq k_0$, where $A_0 \in \mathcal{E}_{k_0}$.*

By the lemma, the restriction of f on A_0 is measure-preserving up to a constant. Thus, if we choose $\mathbf{i}_0 \in \Sigma_4^*$ such that $\Psi_{\mathbf{i}_0}(T) \subset A_0$, then the restriction of f on $\Psi_{\mathbf{i}_0}(T)$ is also measure-preserving up to a constant. Hence, without loss of generality, we assume that $A_0 = \Psi_{\mathbf{i}_0}(T)$ in the sequel.

By Lemmas 2.1 and 2.7, there exists an integer n_1 such that for any $k \in \mathbb{Z}^+$, there exist $\mathbf{j}, \mathbf{j}_1, \dots, \mathbf{j}_p \in \Sigma_4^*$ such that $D_{\mathbf{j}\mathbf{j}_1}, D_{\mathbf{j}\mathbf{j}_2}, \dots, D_{\mathbf{j}\mathbf{j}_p}$ are disjoint and

$$f(\Psi_{\mathbf{i}_0}(E_k)) = \bigcup_{r=1}^p D_{\mathbf{j}\mathbf{j}_r},$$

where each $|\mathbf{j}_r| = n_1$. Furthermore, by Remark 2.1, we can require $D_{\mathbf{j}}$ to be the smallest cylinder containing $f(\Psi_{\mathbf{i}_0}(E_k))$. We denote this \mathbf{j} by $\mathbf{j}'(k)$ and define $\gamma'_k = \sum_{r=1}^p \mu_{\mathbf{j}_r}$. Define

$$\mathcal{M}' = \left\{ \sum_{\mathbf{j} \in \mathcal{A}} \mu_{\mathbf{j}} | \mathcal{A} \subset \{1, \dots, 4\}^{n_1} \right\}.$$

Proof of Theorem 1.2. Notice that $2\mu_1 + \mu_2 + \mu_3 = 1$. In order to prove the lemma, it suffices to show that there exists a polynomial $P(x_1, x_2, x_3)$ with rational coefficients such that $P(\mu_1, \mu_2, \mu_3) = 0$ and $P((1 - x_2 - x_3)/2, x_2, x_3)$ is not identically equal to 0.

Define $\tilde{f} : T \rightarrow D$ by

$$\tilde{f}(x) = f(\Psi_{\mathbf{i}_0}(x)).$$

Let x^* be the unique point in the set $T_2 \cap T_3$. Assume that $\{\tilde{f}(x^*)\} = D_{t_1 t_2 \dots}$.

Given $\mathbf{i} = i_1 i_2 \dots i_k \in \Sigma_4^*$ and $\ell \in \{1, 2, 3, 4\}$, we define $N_\ell(\mathbf{i})$ to be the cardinality of $\{j : i_j = \ell, 1 \leq j \leq k\}$.

Case 1. Assume that there are infinitely many k such that $t_k = 2$. Then $\lim_{q \rightarrow \infty} N_2(\mathbf{j}'(q)) = \infty$. Notice that $\text{card } \mathcal{M}' < +\infty$. Thus, we can choose q_1, q_2 with $1 \leq q_1 < q_2$ such that $\gamma'_{q_1} = \gamma'_{q_2}$ and

$$(2.16) \quad N_2(\mathbf{j}'(q_2)) > N_2(\mathbf{j}'(q_1)).$$

From $\gamma'_{q_1} = \gamma'_{q_2}$ and Lemma 2.9, we have

$$\frac{\mathcal{H}^s(\Psi_{\mathbf{i}_0}(E_{q_2}))}{\mathcal{H}^s(\Psi_{\mathbf{i}_0}(E_{q_1}))} = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_2)})}{\mathcal{H}^s(D_{\mathbf{j}'(q_1)})}.$$

Thus, by (2.16),

$$\mu_1^{q_2 - q_1} = \mu_1^{k_1} \mu_2^{k_2} \mu_3^{k_3}$$

with $k_2 \in \mathbb{Z}^+$ and $k_1, k_3 \in \mathbb{N}$. Since the above equality does not hold if we plug in $\mu_1 = 1/2, \mu_2 = \mu_3 = 0$, we know that μ_2 and μ_3 are algebraic dependent.

Case 2. Assume that there are infinitely many k such that $t_k = 3$. Then using the same method as in Case 1, we can obtain that μ_2 and μ_3 are algebraic dependent.

Case 3. Assume that there are only finitely many k such that $t_k \in \{2, 3\}$. Then there exists $q_0 \in \mathbb{Z}^+$, such that $N_2(\mathbf{j}'(q)) = N_2(\mathbf{j}'(q_0))$ and $N_3(\mathbf{j}'(q)) = N_3(\mathbf{j}'(q_0))$ for all $q \geq q_0$.

By definition,

$$\frac{\mathcal{H}^s(\tilde{f}(E_{q_0}))}{\mathcal{H}^s(E_{q_0})} = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_0)}) \gamma'_{q_0}}{(\mu_2 + \mu_3) \mu_1^{q_0} \cdot \mathcal{H}^s(T)}.$$

Substituting μ_3 by $1 - 2\mu_1 - \mu_2$, we know that γ'_{q_0} is a polynomial of μ_1 and μ_2 with integral coefficients. By using Euclidean algorithm, it is easy to see that there exist polynomials $Q(\mu_1, \mu_2)$ and $R(\mu_2)$ with

rational coefficients, such that $\gamma'_{q_0} = (1 - 2\mu_1)Q(\mu_1, \mu_2) + R(\mu_2)$. Thus $\gamma'_{q_0} = (\mu_2 + \mu_3)Q(\mu_1, \mu_2) + R(\mu_2)$ so that

$$(2.17) \quad \frac{\mathcal{H}^s(\tilde{f}(E_{q_0}))}{\mathcal{H}^s(E_{q_0})} = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_0)})((\mu_2 + \mu_3)Q(\mu_1, \mu_2) + R(\mu_2))}{(\mu_2 + \mu_3)\mu_1^{q_0} \cdot \mathcal{H}^s(T)}.$$

From $E_{q_0} = T_{2[4]^{q_0}} \cup T_{3[1]^{q_0}}$, we know that $\Psi_{\mathbf{i}_0}(T_{2[4]^{q_0}1}) \subset \Psi_{\mathbf{i}_0}(E_{q_0})$. Thus the smallest cylinder of D containing $f(\Psi_{\mathbf{i}_0}(T_{2[4]^{q_0}1}))$ is a subset of the smallest cylinder of D containing $f(\Psi_{\mathbf{i}_0}(E_{q_0}))$. It follows that there exists a polynomial P of μ_1, μ_2, μ_3 with integral coefficients, such that

$$(2.18) \quad \frac{\mathcal{H}^s(\tilde{f}(T_{2[4]^{q_0}1}))}{\mathcal{H}^s(T_{2[4]^{q_0}1})} = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_0)}) \cdot P(\mu_1, \mu_2, \mu_3)}{\mu_1^{q_0+1} \mu_2 \cdot \mathcal{H}^s(T)}.$$

By Lemma 2.9 and the definition of \tilde{f} , the right hand sides of (2.17) and (2.18) are equal so that

$$(2.19) \quad ((\mu_2 + \mu_3)Q(\mu_1, \mu_2) + R(\mu_2))\mu_1\mu_2 = (\mu_2 + \mu_3)P(\mu_1, \mu_2, \mu_3).$$

Case 3.1. Assume that the polynomial $R(x)$ is not identically equal to 0. Then there exists $a \neq 0$ such that $R(a) \neq 0$. Notice that (2.19) does not hold if we plug in $\mu_1 = 1/2$, $\mu_2 = a$, $\mu_3 = -a$. Thus μ_2 and μ_3 are algebraic dependent.

Case 3.2. Assume that the polynomial $R(x)$ is identically equal to 0. Then from (2.17), we have

$$(2.20) \quad \frac{\mathcal{H}^s(\tilde{f}(E_{q_0}))}{\mathcal{H}^s(E_{q_0})} = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_0)}) \cdot Q(\mu_1, \mu_2)}{\mu_1^{q_0} \cdot \mathcal{H}^s(T)}.$$

Since there are only finitely many k such that $t_k = \{2, 3\}$, we can choose $q_3 > q_0$ such that in the right hand side of

$$f(\Psi_{\mathbf{i}_0}(E_{q_3})) = \bigcup_{r=1}^p D_{\mathbf{j}\mathbf{j}_r},$$

where $\mathbf{j}, \mathbf{j}_1, \dots, \mathbf{j}_p$ have same meaning as above, the cylinder $D_{\mathbf{j}\mathbf{j}_r}$ containing $\Psi_{\mathbf{i}_0}(x^*)$ satisfies $N_2(\mathbf{j}_r) = N_3(\mathbf{j}_r) = 0$. It follows that $\gamma'_{q_3} \neq 0$ if we plug in $\mu_1 = 1/2, \mu_2 = \mu_3 = 0$.

Notice that

$$\frac{\mathcal{H}^s(\tilde{f}(E_{q_3}))}{\mathcal{H}^s(E_{q_3})} = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_3)}) \cdot \gamma'_{q_3}}{(\mu_2 + \mu_3)\mu_1^{q_3} \cdot \mathcal{H}^s(T)}.$$

By (2.20) and using Lemma 2.9, we have

$$(\mu_2 + \mu_3)\mu_1^{q_3-q_0}Q(\mu_1, \mu_2) = \frac{\mathcal{H}^s(D_{\mathbf{j}'(q_3)})}{\mathcal{H}^s(D_{\mathbf{j}'(q_0)})} \cdot \gamma'_{q_3}.$$

It follows from $N_2(\mathbf{j}'(q_3)) = N_2(\mathbf{j}'(q_0))$ and $N_3(\mathbf{j}'(q_3)) = N_3(\mathbf{j}'(q_0))$ that $\frac{\mathcal{H}^s(D_{\mathbf{j}'(q_3)})}{\mathcal{H}^s(D_{\mathbf{j}'(q_0)})} = \mu_1^k$ for some $k \in \mathbb{N}$. Thus the above equality does not hold if we plug in $\mu_1 = 1/2, \mu_2 = \mu_3 = 0$ so that μ_2 and μ_3 are algebraic dependent. ■

Example 2.1. Let $n = 4$ and $\Sigma_T = \{2\}$. Let $\mu_2 = e/4$, $\mu_3 = 1/4$ and $\mu_1 = \mu_4 = (1 - \mu_2 - \mu_3)/2$, where e is the Euler constant. Let $\rho_i = \mu_i^2$ for all i . Then $\dim_H D = \dim_H T = 1/2$. By Theorem 1.2, $D \not\sim T$.

3. SUFFICIENT CONDITION FOR $D \sim T$

3.1. Graph-directed sets corresponding to D and T . Now we recall the notion of *graph-directed set*. Let $G = (V, \Gamma)$ be a directed graph and d a positive integer. Suppose for each edge $e \in \Gamma$, there is a corresponding similarity $S_e : \mathbb{R}^d \rightarrow \mathbb{R}^d$ with ratio r_e . Assume that for each vertex $i \in V$, there exists an edge starting from i , and assume that $r_{e_1} \cdots r_{e_k} < 1$ for any cycle $e_1 \cdots e_k$. Then there exists a unique family $\{K_i\}_{i \in V}$ of compact subsets of \mathbb{R}^d such that for any $i \in V$,

$$(3.1) \quad K_i = \bigcup_{j \in V} \bigcup_{e \in \mathcal{E}_{ij}} S_e(K_j),$$

where \mathcal{E}_{ij} is the set of edge starting from i and ending at j . In particular, if the union in (3.1) is disjoint for any i , we call $\{K_i\}_{i \in V}$ are dust-like graph-directed sets on (V, Γ) . For details on graph-directed sets, please see [11, 15].

Similarly as Theorem 2.1 in [13], we have the following lemma which was also pointed out in [20].

Lemma 3.1. *Suppose that $\{K_i\}_{i \in V}$ and $\{K'_i\}_{i \in V}$ are dust-like graph-directed sets on (V, Γ) satisfying (3.1) and $K'_i = \bigcup_{j \in V} \bigcup_{e \in \mathcal{E}_{ij}} S'_e(K'_j)$. If similarities S_e and S'_e have the same ratio for each $e \in \Gamma$, then $K_i \sim K'_i$ for each $i \in V$.*

Definition 3.1. *Assume that $\mathcal{K} = \{K_i\}_{i=1}^m$ and $\mathcal{K}' = \{K'_i\}_{i=1}^m$ are two families of compact subsets of \mathbb{R}^d , where $m \geq 2$ is a given positive integer. We say that two compact subsets A and B of \mathbb{R}^d have same dust-like decomposition w.r.t. \mathcal{K} and \mathcal{K}' , if there exist a positive integer $t \geq 2$ and positive integers $j_1, j_2, \dots, j_t \in \{1, \dots, m\}$, such that*

$$A = \bigcup_{i=1}^t S_i(K_{j_i}), \quad \text{and} \quad B = \bigcup_{i=1}^t S'_i(K'_{j_i}),$$

where the above two unions are disjoint, while for each i , S_i and S'_i are similarities from \mathbb{R}^d to \mathbb{R}^d with same ratio.

Clearly, the following lemma is a weak version of Lemma 3.1.

Lemma 3.2. *Suppose that $\mathcal{K} = \{K_i\}_{i=1}^m$ and $\mathcal{K}' = \{K'_i\}_{i=1}^m$ are two families of compact subsets of \mathbb{R}^d . If for each $1 \leq i \leq m$, K_i and K'_i have same dust-like decomposition w.r.t. \mathcal{K} and \mathcal{K}' , then $K_i \sim K'_i$ for each $i = 1, \dots, m$.*

Given $\mathbf{i} \in \Sigma_n^*$ and a nonnegative integer k . We recall that $L_k(T_{\mathbf{i}}) = \bigcup_{j=1}^{\alpha} T_{\mathbf{i}[1]^{kj}}$ and $R_k(T_{\mathbf{i}}) = \bigcup_{j=n-\beta+1}^n T_{\mathbf{i}[n]^{kj}}$ as defined in (1.2). Now we define

$$L_k(D_{\mathbf{i}}) = \bigcup_{j=1}^{\alpha} D_{\mathbf{i}[1]^{kj}}, \quad R_k(D_{\mathbf{i}}) = \bigcup_{j=n-\beta+1}^n D_{\mathbf{i}[n]^{kj}}.$$

In the rest of this section, we will always assume that $\log \rho_1 / \log \rho_n \in \mathbb{Q}$ and every touching letter is substitutable. Thus, for any $i \in \Sigma_T$, there exist $\mathbf{j}_i \in \Sigma_n^*$ and $k_i, k'_i \in \mathbb{N}$, such that one of the following holds:

- (1) $\text{diam } L_{k_i}(T_{i+1}) = \text{diam } L_{k'_i}(T_{i\mathbf{j}_i})$ and the last letter of \mathbf{j}_i does not belong to $\{1\} \cup (\Sigma_T + 1)$,
- (2) $\text{diam } R_{k_i}(T_i) = \text{diam } R_{k'_i}(T_{(i+1)\mathbf{j}_i})$ and the last letter of \mathbf{j}_i does not belong to $\{n\} \cup \Sigma_T$.

From $\log \rho_1 / \log \rho_n \in \mathbb{Q}$, there exist $p, q \in \mathbb{Z}^+$ such that $\rho_1^p = \rho_n^q$. Notice that we can choose p and q large enough such that $p, q > \max\{k'_i + |\mathbf{j}_i| : i \in \Sigma_T\}$. As a result, we have

$$(3.2) \quad L_{k'_i}(T_{i[n]^{2q}\mathbf{j}_i}) \cap R_{3q}(T_i) = L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i}) \cap R_{3q}(D_i) = \emptyset \quad \text{and} \quad L_{k'_i}(D_{i\mathbf{j}_i}) \cap R_q(D_i) = \emptyset$$

for any left substitutable touching letter i , and

$$(3.3) \quad L_{3p}(T_{i+1}) \cap R_{k'_i}(T_{(i+1)[1]^{2p}\mathbf{j}_i}) = L_{3p}(D_{i+1}) \cap R_{k'_i}(D_{(i+1)[1]^{2p}\mathbf{j}_i}) = \emptyset \quad \text{and} \quad L_p(D_{i+1}) \cap R_{k'_i}(D_{(i+1)\mathbf{j}_i}) = \emptyset$$

for any right substitutable touching letter i . We remark that this restriction is useful in the definition of $D_i^{(4)}$ and in the proof of Lemma 3.11. We will fix p, q in this section.

Given a positive integer m . Let $\mathcal{J}_m = \bigcup_{|\mathbf{i}|=m} I_{\mathbf{i}}$ and $J_{m,1}, \dots, J_{m,c_m}$ are the connected components of \mathcal{J}_m , spaced from left to right. We define $\Lambda_i = \{j \mid I_j \subset J_{1,i}\}$ for $i = 1, \dots, c_1$. Define

$$T_i^{(1)} = \bigcup_{j \in \Lambda_i} T_j, \quad D_i^{(1)} = \bigcup_{j \in \Lambda_i} D_j, \quad \forall i = 1, \dots, c_1.$$

Then for any $\mathbf{i} \in \Sigma_n^*$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} L_k(T_{\mathbf{i}}) &= \Psi_{\mathbf{i}[1]^k}(T_1^{(1)}), & R_k(T_{\mathbf{i}}) &= \Psi_{\mathbf{i}[n]^k}(T_{c_1}^{(1)}), \\ L_k(D_{\mathbf{i}}) &= \Phi_{\mathbf{i}[1]^k}(D_1^{(1)}), & R_k(D_{\mathbf{i}}) &= \Phi_{\mathbf{i}[n]^k}(D_{c_1}^{(1)}). \end{aligned}$$

For any touching letter i , we define

$$\begin{aligned} T_i^{(2)} &= R_0(T_i) \cup L_0(T_{i+1}), & D_i^{(2)} &= R_0(D_i) \cup L_0(D_{i+1}), \\ T_i^{(3)} &= R_q(T_i) \cup L_p(T_{i+1}), & D_i^{(3)} &= R_q(D_i) \cup L_p(D_{i+1}). \end{aligned}$$

We remark that $R_0(T_i) = \Psi_i(T_{c_1}^{(1)})$ and $L_0(T_{i+1}) = \Psi_{i+1}(T_1^{(1)})$. Furthermore, for any touching letter i , if i is left substitutable, we define

$$T_i^{(4)} = R_q(T_i) \cup L_{k_i}(T_{i+1}), \quad D_i^{(4)} = L_{k'_i}(D_{i\mathbf{j}_i}) \cup R_q(D_i).$$

Otherwise, we define

$$T_i^{(4)} = R_{k_i}(T_i) \cup L_p(T_{i+1}), \quad D_i^{(4)} = L_p(D_{i+1}) \cup R_{k'_i}(D_{(i+1)\mathbf{j}_i}).$$

Define $\mathcal{T} = \{T\} \cup \{T_i^{(1)} : i = 1, \dots, c_1\} \cup \{T_i^{(j)} : i \in \Sigma_T, j = 2, 3, 4\}$ and $\mathcal{D} = \{D\} \cup \{D_i^{(1)} : i = 1, \dots, c_1\} \cup \{D_i^{(j)} : i \in \Sigma_T, j = 2, 3, 4\}$. We will show that each corresponding pair in \mathcal{T} and \mathcal{D} have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .

3.2. The family \mathcal{T}^* . Given $\mathbf{i} \in \Sigma_n^*$. We say that $\mathbf{i} \in \Sigma_L^*$ if there exist $\mathbf{i}' \in \Sigma_n^* \cup \{\emptyset\}$, $j \in \Sigma_T + 1$ and $k \in \mathbb{N}$ such that $\mathbf{i} = \mathbf{i}'j[1]^k$. Similarly, we say that $\mathbf{i} \in \Sigma_R^*$ if there exist $\mathbf{i}' \in \Sigma_n^* \cup \{\emptyset\}$, $j \in \Sigma_T$ and $k \in \mathbb{N}$ such that $\mathbf{i} = \mathbf{i}'j[n]^k$. The following lemma is easy to check.

Lemma 3.3. *Given $\mathbf{i} \in \Sigma_n^*$ and $j \in \{1, \dots, c_1\}$. Then $\Psi_{\mathbf{i}}(T_j^{(1)})$ is T -separate if and only if one of the following conditions holds: (1). $2 \leq j \leq c_1 - 1$; (2). $j = 1$ and $\mathbf{i} \notin \Sigma_L^*$; (3). $j = c_1$ and $\mathbf{i} \notin \Sigma_R^*$.*

Define

$$\mathcal{T}^{(1)} = \{\Psi_{\mathbf{i}}(T_j^{(1)}) : \mathbf{i} \in \Sigma_n^* \cup \{\emptyset\} \text{ and } j \in \{1, \dots, c_1\} \text{ such that } \Psi_{\mathbf{i}}(T_j^{(1)}) \text{ is } T\text{-separate}\},$$

$$\mathcal{T}^{(k)} = \{\Psi_{\mathbf{i}}(T_j^{(k)}) : \mathbf{i} \in \Sigma_n^* \cup \{\emptyset\} \text{ and } j \in \Sigma_T\}, \quad k = 2, 3.$$

It is clear that all sets in $\mathcal{T}^{(2)}$ and $\mathcal{T}^{(3)}$ are T -separate. Define

$$\mathcal{T}^* = \{A \mid A \text{ is a disjoint union of finitely many } (\geq 2) \text{ sets in the class } \mathcal{T}^{(1)} \cup \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}\}.$$

Remark 3.1. Assume that $A \in \mathcal{T}^{(1)}$ with $A \subset (0, 1)$. Then it is easy to check that $\Psi_{\mathbf{i}}(A) \in \mathcal{T}^{(1)}$ for any $\mathbf{i} \in \Sigma_n^*$. It follows that $\Psi_{\mathbf{i}}(B) \in \mathcal{T}^*$ for all $B \in \mathcal{T}^*$ with $B \subset (0, 1)$ and all $\mathbf{i} \in \Sigma_n^*$.

Let $\Sigma_n^\infty = \{i_1 i_2 \dots i_m \dots \mid i_m \in \{1, \dots, n\} \text{ for all } m\}$ as defined in the proof of Lemma 2.6. Given $\mathbf{i} = i_1 \dots i_m \dots \in \Sigma_n^\infty$, there exists a unique point $x \in T$ such that

$$\{x\} = \bigcap_{m=1}^{\infty} \Psi_{i_1 \dots i_m}([0, 1]).$$

We denote this unique x by $\pi_T(\mathbf{i})$. Then $\pi_T : \Sigma_n^\infty \rightarrow T$ is a surjection. Similarly, we can define $\pi_D : \Sigma_n^\infty \rightarrow D$ by

$$\{\pi_D(\mathbf{i})\} = \bigcap_{m=1}^{\infty} \Phi_{i_1 \dots i_m}([0, 1]), \quad \forall \mathbf{i} = i_1 \dots i_m \dots \in \Sigma_n^\infty.$$

Since D is dust-like, π_D is a bijection.

By definition of π_T and π_D , it is easy to check that

$$(3.4) \quad \pi_D \circ \pi_T^{-1}(\Psi_{\mathbf{i}}(T_j^{(k)})) = \Phi_{\mathbf{i}}(D_j^{(k)}), \quad \forall \Psi_{\mathbf{i}}(T_j^{(k)}) \in \mathcal{T}^{(k)}, \quad k = 1, 2, 3.$$

Using this fact, we have the following lemma.

Lemma 3.4. *Let $A \in \mathcal{T}^*$. Then A and $\pi_D \circ \pi_T^{-1}(A)$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .*

Proof. Assume that $A = \bigcup_{i=1}^m A_i$, where $m \geq 2$, $A_i \in \mathcal{T}^{(1)} \cup \mathcal{T}^{(2)} \cup \mathcal{T}^{(3)}$ and the union is disjoint. Then

$$(3.5) \quad \pi_D \circ \pi_T^{-1}(A) = \bigcup_{i=1}^m \pi_D \circ \pi_T^{-1}(A_i).$$

From the union in $\bigcup_{i=1}^m A_i$ is disjoint, we can see that the union in $\bigcup_{i=1}^m \pi_T^{-1}(A_i)$ is disjoint. Since π_D is a bijection, we know that the union in (3.5) is also disjoint. From (3.4), we can see that the lemma holds. \blacksquare

3.3. Graph-directed decomposition of \mathcal{T} and \mathcal{D} and the proof of sufficient condition. The following lemma is easy to show.

Lemma 3.5. *The following pairs have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .*

- (i) T and D ;
- (ii) $T_i^{(1)}$ and $D_i^{(1)}$ for $i = 1, \dots, c_1$;

Proof. (i) Clearly, T and D can be decomposed to following disjoint unions.

$$T = \bigcup_{i=1}^{c_1} T_i^{(1)}, \quad D = \bigcup_{i=1}^{c_1} D_i^{(1)}.$$

(ii) Given $i = 1, \dots, c_1$. Notice that

$$(3.6) \quad T_i^{(1)} = \bigcup_{j \in \Lambda_i} T_j = \bigcup_{j \in \Lambda_i} \Psi_j(T) = \bigcup_{j \in \Lambda_i} \bigcup_{k=1}^{c_1} \Psi_j(T_k^{(1)}).$$

Let $b(i)$ and $e(i)$ be the minimal and maximal element in Λ_i , respectively. If $b(i) = e(i)$, then

$$T_i^{(1)} = \bigcup_{k=1}^{c_1} \Psi_{b(i)}(T_k^{(1)}) \in \mathcal{T}^*,$$

since each $\Psi_{b(i)}(T_k^{(1)})$ is T -separate in this case. If $b(i) < e(i)$, then for any $b(i) \leq j < e(i)$,

$$\Psi_j(T_{c_1}^{(1)}) \cup \Psi_{j+1}(T_1^{(1)}) = T_j^{(2)},$$

and other $\Psi_j(T_k^{(1)})$ in (3.6) are T -separate so that they belong to $\mathcal{T}^{(1)}$. Thus $T_i^{(1)}$ also belongs to \mathcal{T}^* in this case. By Lemma 3.4, $T_i^{(1)}$ and $D_i^{(1)} = \pi_D \circ \pi_T^{-1}(T_i^{(1)})$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . \blacksquare

Remark 3.2. It follows from the above lemma that $\mathcal{T}^{(1)} \subset \mathcal{T}^*$.

Lemma 3.6. *Given $\mathbf{i} \in \Sigma_n^*$ and two nonnegative integers u, v with $u < v$. The pairs $L_u(T_{\mathbf{i}}) \setminus L_v(T_{\mathbf{i}})$ and $L_u(D_{\mathbf{i}}) \setminus L_v(D_{\mathbf{i}})$, $R_u(T_{\mathbf{i}}) \setminus R_v(T_{\mathbf{i}})$ and $R_u(D_{\mathbf{i}}) \setminus R_v(D_{\mathbf{i}})$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} , respectively.*

Proof. Without loss of generality, we only show that $L_u(T_{\mathbf{i}}) \setminus L_v(T_{\mathbf{i}})$ and $L_u(D_{\mathbf{i}}) \setminus L_v(D_{\mathbf{i}})$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . Clearly,

$$\pi_D \circ \pi_T^{-1}(L_u(T_{\mathbf{i}}) \setminus L_v(T_{\mathbf{i}})) = L_u(D_{\mathbf{i}}) \setminus L_v(D_{\mathbf{i}}).$$

Thus, from Lemma 3.4 and noticing that $L_u(T_{\mathbf{i}}) \setminus L_v(T_{\mathbf{i}}) = \bigcup_{k=u}^{v-1} (L_k(T_{\mathbf{i}}) \setminus L_{k+1}(T_{\mathbf{i}}))$, it suffices to show that $L_k(T_{\mathbf{i}}) \setminus L_{k+1}(T_{\mathbf{i}}) \in \mathcal{T}^*$ for all $k \in \mathbb{N}$.

Given $k \in \mathbb{N}$. Assume that $1 \notin \Sigma_T$, i.e. $\alpha = 1$. Then

$$L_k(T_{\mathbf{i}}) \setminus L_{k+1}(T_{\mathbf{i}}) = \bigcup_{j=2}^n T_{\mathbf{i}[1]^{k+1}j} = \bigcup_{j=2}^{c_1} \Psi_{\mathbf{i}[1]^{k+1}}(T_j^{(1)}).$$

Notice that $\Psi_{\mathbf{i}[1]^{k+1}}(T_{c_1}^{(1)})$ is T -separate in this case. By Remark 3.2, $L_k(T_{\mathbf{i}}) \setminus L_{k+1}(T_{\mathbf{i}}) \in \mathcal{T}^*$.

Assume that $1 \in \Sigma_T$, i.e. $\alpha \geq 2$. Then $L_k(T_{\mathbf{i}}) \setminus L_{k+1}(T_{\mathbf{i}}) = \Psi_{\mathbf{i}[1]^k}(A)$, where

$$(3.7) \quad A = \left(\bigcup_{j=\alpha+1}^n T_{1j} \right) \cup \left(\bigcup_{\ell=2}^{\alpha} T_{\ell} \right) = \left(\bigcup_{j=2}^{c_1} \Psi_1(T_j^{(1)}) \right) \cup \left(\bigcup_{\ell=2}^{\alpha} \bigcup_{j=1}^{c_1} \Psi_{\ell}(T_j^{(1)}) \right).$$

For each $1 \leq \ell \leq \alpha - 1$, $\Psi_{\ell}(T_{c_1}^{(1)}) \cup \Psi_{\ell+1}(T_1^{(1)}) = T_{\ell}^{(2)}$. Furthermore, other $\Psi_1(T_j^{(1)})$ and $\Psi_{\ell}(T_j^{(1)})$ in the right-hand side of (3.7) are T -separate. By Remark 3.1, it is easy to see that $L_k(T_{\mathbf{i}}) \setminus L_{k+1}(T_{\mathbf{i}}) \in \mathcal{T}^*$. ■

From this fact, we have the following lemma.

Lemma 3.7. *For any $i \in \Sigma_T$, $T_i^{(2)}$ and $D_i^{(2)}$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .*

Proof. For each touching letter i , we have following disjoint unions.

$$\begin{aligned} T_i^{(2)} &= (R_0(T_i) \setminus R_q(T_i)) \cup (L_0(T_{i+1}) \setminus L_p(T_{i+1})) \cup T_i^{(3)}, \quad \text{and} \\ D_i^{(2)} &= (R_0(D_i) \setminus R_q(D_i)) \cup (L_0(D_{i+1}) \setminus L_p(D_{i+1})) \cup D_i^{(3)}. \end{aligned}$$

The lemma follows from Lemma 3.6. ■

Given $\mathbf{i} = i_1 i_2 \dots i_m, \mathbf{j} = j_1 j_2 \dots j_m \in \Sigma_n^*$ with the same length. We denote by $\mathbf{i} < \mathbf{j}$ if there exists $1 \leq k \leq m$ such that $i_k < j_k$ and $i_t = j_t$ for $1 \leq t < k$. We denote by $\mathbf{i} \leq \mathbf{j}$ if $\mathbf{i} < \mathbf{j}$ or $\mathbf{i} = \mathbf{j}$.

Given $\mathbf{i}, \mathbf{j} \in \Sigma_n^*$ with $\mathbf{i} < \mathbf{j}$. We say that (\mathbf{i}, \mathbf{j}) is a *joint pair* if $\mathbf{k} \leq \mathbf{i}$ for every $\mathbf{k} \in \Sigma_n^*$ with $\mathbf{k} < \mathbf{j}$.

Lemma 3.8. *Let $\mathbf{i}, \mathbf{j} \in \Sigma_n^*$ with $\Psi_{\mathbf{i}}(0) < \Psi_{\mathbf{j}}(1)$. Suppose that $[\Psi_{\mathbf{i}}(0), \Psi_{\mathbf{j}}(1)] \cap T$ is T -separate and $\max\{|\mathbf{i}|, |\mathbf{j}|\} \leq \min\{p, q\}$. Then $[\Psi_{\mathbf{i}}(0), \Psi_{\mathbf{j}}(1)] \cap T \in \mathcal{T}^*$.*

Proof. Given $\mathbf{k} \in \Sigma_n^*$, we define the middle part of $T_{\mathbf{k}}$ to be $M(T_{\mathbf{k}}) = \bigcup_{j=\alpha+1}^{n-\beta} T_{kj}$. It is clear that $M(T_{\mathbf{k}}) = T_{\mathbf{k}} \setminus \{L_0(T_{\mathbf{k}}) \cup R_0(T_{\mathbf{k}})\}$. We remark that $M(T_{\mathbf{k}}) = \emptyset$ for all \mathbf{k} if $\alpha = n - \beta$. In case that $\alpha < n - \beta$, it is clear that $M(T_{\mathbf{k}}) \in \mathcal{T}^*$ for all \mathbf{k} .

Without loss of generality, we may assume that $|\mathbf{i}| \leq |\mathbf{j}|$. Define $k = |\mathbf{j}| - |\mathbf{i}|$. Then $[\Psi_{\mathbf{i}[1]^k}(0), \Psi_{\mathbf{j}}(1)] \cap T$ is T -separate since $\Psi_{\mathbf{i}[1]^k}(0) = \Psi_{\mathbf{i}}(0)$. Thus, noticing that the lemma holds in case that $\mathbf{i} = \mathbf{j}$, we assume that $\mathbf{i} < \mathbf{j}$ in the sequel of the proof.

Define $m = |\mathbf{i}|$. Let $\Sigma(\mathbf{i}, \mathbf{j}) = \{\mathbf{k} \in \Sigma_n^m : \mathbf{i} \leq \mathbf{k} \leq \mathbf{j}\}$. Then

$$(3.8) \quad [\Psi_{\mathbf{i}}(0), \Psi_{\mathbf{j}}(1)] \cap T = \bigcup_{\mathbf{k} \in \Sigma(\mathbf{i}, \mathbf{j})} T_{\mathbf{k}}.$$

Now we arbitrary pick a joint pair (\mathbf{u}, \mathbf{v}) with $\mathbf{u}, \mathbf{v} \in \Sigma(\mathbf{i}, \mathbf{j})$. Notice that

$$T_{\mathbf{u}} = L_0(T_{\mathbf{u}}) \cup R_0(T_{\mathbf{u}}) \cup M(T_{\mathbf{u}}), \quad T_{\mathbf{v}} = L_0(T_{\mathbf{v}}) \cup R_0(T_{\mathbf{v}}) \cup M(T_{\mathbf{v}}).$$

In case that $R_0(T_{\mathbf{u}})$ is T -separate, we have $R_0(T_{\mathbf{u}}) \in \mathcal{T}^{(1)}$. Also, in this case, we must have $L_0(T_{\mathbf{v}})$ is T -separate so that $L_0(T_{\mathbf{v}}) \in \mathcal{T}^{(1)}$. It follows that there exists $A(\mathbf{u}, \mathbf{v}) \in \mathcal{T}^*$ such that

$$(3.9) \quad T_{\mathbf{u}} \cup T_{\mathbf{v}} = L_0(T_{\mathbf{u}}) \cup R_0(T_{\mathbf{v}}) \cup A(\mathbf{u}, \mathbf{v}),$$

where the union is disjoint.

In case that $R_0(T_{\mathbf{u}})$ is not T -separate, we define s to be the maximal nonnegative integer which satisfies $\mathbf{u} = \mathbf{u}'[n]^s$ for some $\mathbf{u}' \in \Sigma_n^*$. Let $\mathbf{u}' = u_1 u_2 \cdots u_{m-s}$. From Lemma 3.3, we have $u_{m-s} \in \Sigma_T$ so that $\mathbf{v} = \mathbf{v}'[1]^s$ where $\mathbf{v}' = u_1 \cdots u_{m-s-1}(u_{m-s} + 1)$. It is clear that $s < \min\{p, q\}$ since $|\mathbf{u}| = |\mathbf{i}| \leq \min\{p, q\}$. Notice that

$$R_0(T_{\mathbf{u}}) = R_{q-s}(T_{\mathbf{u}}) \cup \left(R_0(T_{\mathbf{u}}) \setminus R_{q-s}(T_{\mathbf{u}}) \right), \quad L_0(T_{\mathbf{v}}) = L_{p-s}(T_{\mathbf{v}}) \cup \left(L_0(T_{\mathbf{v}}) \setminus L_{p-s}(T_{\mathbf{v}}) \right),$$

where the unions are disjoint and $R_0(T_{\mathbf{u}}) \setminus R_{q-s}(T_{\mathbf{u}}), L_0(T_{\mathbf{v}}) \setminus L_{p-s}(T_{\mathbf{v}}) \in \mathcal{T}^*$ by Lemma 3.6. Since

$$\begin{aligned} R_{q-s}(T_{\mathbf{u}}) \cup L_{p-s}(T_{\mathbf{v}}) &= R_{q-s}(T_{\mathbf{u}'[n]^s}) \cup L_{p-s}(T_{\mathbf{v}'[1]^s}) = R_q(T_{\mathbf{u}'}) \cup L_p(T_{\mathbf{v}'}), \\ &= \Psi_{u_1 \cdots u_{m-s-1}} \left(R_q(T_{u_{m-s}}) \cup L_p(T_{u_{m-s}+1}) \right) = \Psi_{u_1 \cdots u_{m-s-1}} (T_{k_{m-s}}^{(3)}) \in \mathcal{T}^{(3)}, \end{aligned}$$

we know that in this case, there also exists $A(\mathbf{u}, \mathbf{v}) \in \mathcal{T}^*$ such that (3.9) holds while the union is disjoint.

Notice that $L_0(T_{\mathbf{i}})$ and $R_0(T_{\mathbf{j}})$ are T -separate so that they are all in \mathcal{T}^* . Using (3.8) and (3.9), we can see that the lemma holds. ■

Corollary 3.1. *Given $i = 1, 2, \dots, n$, $k \in \mathbb{N}$ and $\mathbf{j} \in \Sigma_n^*$ with $k + |\mathbf{j}| < \min\{p, q\}$. Assume that the last letter of \mathbf{j} does not belong to $\{1\} \cup (\Sigma_T + 1)$. Then $R_q(T_i) \setminus \left(R_{3q}(T_i) \cup L_k(T_{i[n]^{2q}\mathbf{j}}) \right) \in \mathcal{T}^*$.*

Proof. Let $\mathbf{j} = j_1 j_2 \cdots j_m$. Then $j_m > 1$. Define $\mathbf{u} = j_1 \cdots j_{m-1}(j_m - 1)$. It is easy to check that

$$R_q(T_i) \setminus \left(R_{3q}(T_i) \cup L_k(T_{i[n]^{2q}\mathbf{j}}) \right) = \left([a_1, b_1] \cup [a_2, b_2] \right) \cap T,$$

where $a_1 = \Psi_{i[n]^q(n-\beta+1)}(0)$, $b_1 = \Psi_{i[n]^{2q}\mathbf{u}}(1)$, $a_2 = \Psi_{i[n]^{2q}\mathbf{j}[1]^k(\alpha+1)}(0)$, $b_2 = \Psi_{i[n]^{3q}(n-\beta)}(1)$. Thus it suffices to show that $[a_1, b_1] \cap T \in \mathcal{T}^*$ and $[a_2, b_2] \cap T \in \mathcal{T}^*$. Notice that

$$[a_1, b_1] \cap T = \left(R_q(T_i) \setminus R_{2q-1}(T_i) \right) \cup \left([a'_1, b_1] \cap T \right),$$

where $a'_1 = \Psi_{i[n]^{2q-1}(n-\beta+1)}(0)$ and the union is disjoint. Using Lemma 3.8, we have

$$[a'_1, b_1] \cap T = \Psi_{i[n]^{2q-1}} \left([\Psi_{(n-\beta+1)}(0), \Psi_{n\mathbf{u}}(1)] \cap T \right) \in \mathcal{T}^*$$

so that $[a_1, b_1] \cap T \in \mathcal{T}^*$.

Let s be the maximal nonnegative integer such that $\mathbf{j} = [n]^s \mathbf{j}'$ for some $\mathbf{j}' \in \Sigma_n^*$. Then $s \leq |\mathbf{j}| - 1 < q - 1$. Notice that

$$[a_2, b_2] \cap T = \left(R_{2q+s+1}(T_i) \setminus R_{3q}(T_i) \right) \cup \left([a_2, b'_2] \cap T \right),$$

where $b'_2 = \Psi_{i[n]^{2q+s+1}(n-\beta)}(1)$ and the union is disjoint. From $|\mathbf{j}'| + k + 1 \leq \min\{p, q\}$ and Lemma 3.8, we have

$$[a_2, b'_2] \cap T = \Psi_{i[n]^{2q+s}} \left([\Psi_{\mathbf{j}'[1]^k(\alpha+1)}(0), \Psi_{n(n-\beta)}(1)] \cap T \right) \in \mathcal{T}^*$$

so that $[a_2, b_2] \cap T \in \mathcal{T}^*$. ■

The following lemma is useful in the proof of Lemma 3.11.

Lemma 3.9. *For any left substitutable touching letter i , we have*

$$(3.10) \quad \Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(T_i^{(4)}) = R_{3q}(T_i) \cup L_{2p+k_i}(T_{i+1}),$$

$$(3.11) \quad \Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(D_i^{(4)}) = L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i}) \cup R_{3q}(D_i).$$

Proof. By definition of $T_i^{(4)}$, in order to prove (3.10), it suffices to show that

$$\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(R_q(T_i)) = R_{3q}(T_i) \quad \text{and} \quad \Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(L_{k_i}(T_{i+1})) = L_{2p+k_i}(T_{i+1}).$$

It is clear that

$$\begin{aligned} \text{diam } \Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(R_q(T_i)) &= \text{diam } R_{3q}(T_i) \quad \text{and} \\ \text{diam } \Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(L_{k_i}(T_{i+1})) &= \text{diam } L_{2p+k_i}(T_{i+1}). \end{aligned}$$

Notice that the maximum value of $\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(R_q(T_i))$ is

$$\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(\Psi_i(1)) = \Psi_{i[n]^{2q}}(1) = \Psi_i(1),$$

which equals the maximum value of $R_{3q}(T_i)$. It follows that $\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(R_q(T_i)) = R_{3q}(T_i)$.

Since i is a touching letter, the minimum value of T_{i+1} equals $\Psi_i(1)$. Thus the minimum value of $\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(L_{k_i}(T_{i+1}))$ is also $\Psi_i(1)$, which is equals the minimum value of $L_{2p+k_i}(T_{i+1})$. It follows that $\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(L_{k_i}(T_{i+1})) = L_{2p+k_i}(T_{i+1})$.

In order to prove (3.11), it suffices to show that

$$\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(L_{k'_i}(D_{i\mathbf{j}_i})) = L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i}) \quad \text{and} \quad \Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(R_q(D_i)) = R_{3q}(D_i).$$

Similarly as above, we can easily see that $\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(R_q(D_i)) = R_{3q}(D_i)$. Using diameter and noticing that the minimum value of $\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(L_{k'_i}(D_{i\mathbf{j}_i}))$ is

$$\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(\Phi_{i\mathbf{j}_i}(0)) = \Phi_{i[n]^{2q}\mathbf{j}_i}(0),$$

which equals the minimum value of $L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i})$, we know that $\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(L_{k'_i}(D_{i\mathbf{j}_i})) = L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i})$. ■

The following lemma is natural.

Lemma 3.10. *Given $\mathbf{i}, \mathbf{j} \in \Sigma_n^*$. If there exist $u, v \in \mathbb{Z}^+$ such that $L_u(T_{\mathbf{i}})$ is T -separate and $\text{diam } L_u(T_{\mathbf{i}}) = \text{diam } L_v(T_{\mathbf{j}})$, then $L_u(T_{\mathbf{i}})$ and $L_v(D_{\mathbf{j}})$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .*

Proof. It is clear that

$$L_u(T_{\mathbf{i}}) = \bigcup_{k=1}^{\alpha} T_{\mathbf{i}[1]^u k} = \Psi_{\mathbf{i}[1]^u}(T_1^{(1)}), \quad L_v(D_{\mathbf{j}}) = \bigcup_{k=1}^{\alpha} D_{\mathbf{j}[1]^v k} = \Phi_{\mathbf{j}[1]^v}(D_1^{(1)}).$$

By $\text{diam } L_u(T_{\mathbf{i}}) = \text{diam } L_v(T_{\mathbf{j}})$, we have $\rho_1^u \rho_{\mathbf{i}} = \rho_1^v \rho_{\mathbf{j}}$ so that the contraction ratios of $\Phi_{\mathbf{i}[1]^u}$ and $\Phi_{\mathbf{j}[1]^v}$ are same. Thus the lemma follows from that $T_1^{(1)}$ and $D_1^{(1)}$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . ■

Based on the above lemmas, now we can prove the following crucial lemma.

Lemma 3.11. *For any $i \in \Sigma_T$, the pairs $T_i^{(3)}$ and $D_i^{(3)}$, $T_i^{(4)}$ and $D_i^{(4)}$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .*

Proof. Without loss of generality, we only show that the lemma holds for every left substitutable touching letter i . By Lemma 3.9, we have

$$(3.12) \quad T_i^{(3)} = A_1 \cup \left(L_p(T_{i+1}) \setminus L_{2p+k_i}(T_{i+1}) \right) \cup A_2 \cup \left(\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(T_i^{(4)}) \right),$$

$$(3.13) \quad D_i^{(3)} = B_1 \cup \left(L_p(D_{i+1}) \setminus L_{2p+k_i}(D_{i+1}) \right) \cup B_2 \cup \left(\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(D_i^{(4)}) \right), \quad \text{where}$$

$$\begin{aligned} A_1 &= R_q(T_i) \setminus \left(L_{k'_i}(T_{i[n]^{2q}\mathbf{j}_i}) \cup R_{3q}(T_i) \right), & A_2 &= L_{k'_i}(T_{i[n]^{2q}\mathbf{j}_i}), \\ B_1 &= R_q(D_i) \setminus \left(L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i}) \cup R_{3q}(D_i) \right), & B_2 &= L_{2p+k_i}(D_{i+1}). \end{aligned}$$

Notice that $L_{k'_i}(D_{i[n]^{2q}\mathbf{j}_i}) \cap R_{3q}(D_i) = \emptyset$ by (3.2). Since D is dust-like, it is clear that the union in (3.13) is disjoint. By definition, the last letter of \mathbf{j}_i does not belong to $\{1\} \cup (\Sigma_T + 1)$. Thus, using Lemma 3.3, we know that $A_2 = L_{k'_i}(T_{i[n]^{2q}\mathbf{j}_i}) = \Psi_{i[n]^{2q}\mathbf{j}_i[1]^{k'_i}}(T_1^{(1)})$ is T -separate. Hence, by $L_{k'_i}(T_{i[n]^{2q}\mathbf{j}_i}) \cap R_{3q}(T_i) = \emptyset$, we know that the union in (3.12) is disjoint.

By Lemma 3.10, we know that $L_p(T_{i+1}) \setminus L_{2p+k_i}(T_{i+1})$ and $L_p(D_{i+1}) \setminus L_{2p+k_i}(D_{i+1})$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . Thus in order to show that $T_i^{(3)}$ and $D_i^{(3)}$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} , it suffices to show that A_1 and B_1 , A_2 and B_2 have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .

Notice that $|\mathbf{j}_i| + k'_i < \min\{p, q\}$ and $L_{k'_i}(T_{i[n]^{2q}\mathbf{j}_i})$ is T -separate. By Corollary 3.1, we have $A_1 \in \mathcal{T}^*$. It is clear that $B_1 = \pi_D \circ \pi_T^{-1}(A_1)$. By Lemma 3.4, A_1 and B_1 have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .

Notice that $\text{diam } A_2 = \rho_n^{2q} \text{diam } L_{k'_i}(T_{i\mathbf{j}_i})$ and $\text{diam } L_{2p+k_i}(T_{i+1}) = \rho_1^{2p} \text{diam } L_{k_i}(T_{i+1})$. From the definition of k_i, k'_i and \mathbf{j}_i , we know that $\text{diam } L_{k'_i}(T_{i\mathbf{j}_i}) = \text{diam } L_{k_i}(T_{i+1})$ so that $\text{diam } A_2 = \text{diam } L_{2p+k_i}(T_{i+1})$. Since A_2 is T -separate, by Lemma 3.10, A_2 and B_2 have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . Hence, $T_i^{(3)}$ and $D_i^{(3)}$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .

Using Lemma 3.9 again,

$$\begin{aligned} T_i^{(4)} &= A_1 \cup A_3 \cup A_2 \cup \left(\Psi_{i[n]^{2q}} \circ \Psi_i^{-1}(T_i^{(4)}) \right), \\ D_i^{(4)} &= B_1 \cup B_3 \cup B'_2 \cup \left(\Phi_{i[n]^{2q}} \circ \Phi_i^{-1}(D_i^{(4)}) \right), \quad \text{where} \end{aligned}$$

$$A_3 = L_{k_i}(T_{i+1}) \setminus L_{2p+k_i}(T_{i+1}), \quad B_3 = L_{k'_i}(D_{i\mathbf{j}_i}) \setminus L_{2p+k'_i}(D_{i\mathbf{j}_i}), \quad B'_2 = L_{2p+k'_i}(D_{i\mathbf{j}_i}).$$

Similarly as above, we can see that A_2 and B'_2 have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} .

By Lemma 3.10, A_3 and $L_{k_i}(D_{i+1}) \setminus L_{2p+k_i}(D_{i+1})$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . Notice that

$$\begin{aligned} L_{k_i}(D_{i+1}) \setminus L_{2p+k_i}(D_{i+1}) &= \Phi_{(i+1)[1]^{k_i}} \left(D_1^{(1)} \setminus \Phi_{[1]^{2p}}(D_1^{(1)}) \right), \\ B_3 &= \Phi_{i\mathbf{j}_i[1]^{k'_i}} \left(D_1^{(1)} \setminus \Phi_{[1]^{2p}}(D_1^{(1)}) \right). \end{aligned}$$

By the definition of k_i, k'_i and \mathbf{j}_i , we know that $\rho_{i+1}\rho_1^{k_i} = \rho_1^{k'_i}\rho_{i\mathbf{j}_i}$. Thus it is easy to see A_3 and B_3 have same decomposition w.r.t. \mathcal{T} and \mathcal{D} . As a result, $T_i^{(4)}$ and $D_i^{(4)}$ have same dust-like decomposition w.r.t. \mathcal{T} and \mathcal{D} . ■

Proof of Theorem 1.3 The theorem follows from Lemmas 3.2, 3.5, 3.7 and 3.11. ■

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Department of Mathematics, Zhejiang University, Hangzhou, 310027, China,
ruanhj@zju.edu.cn.

Department of Mathematics, Michigan State University, East Lansing MI, 48824, USA,
ywang@math.msu.edu.

Institute of Mathematics, Zhejiang Wanli University, Ningbo, 315100, China,
xilf@zwu.edu.cn.